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Collusion and signaling in auctions with interdependent values ☆

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Abstract

The standard approach to collusion in auctions posits an uninformed, disinterested third party who, prior to the auction, designs and implements a collusive mechanism. In environments where collusion agreements are likely to be both proposed and executed by the involved parties, this approach may neglect information leakages from simply a proposal to collude. We consider a model of collusion where one informed bidder proposes to another, as in Eső and Schummer (2004). We allow for general interdependent values and affiliated signals. In contrast to third party modeling approaches, collusion is inefficient from both a social perspective (except for the case of pure common values) and from the perspective of the ring. Both bidders are better off than without colluding, but the surplus extracted from the seller is distributed asymmetrically between the bidders. The potential for information leakage from proposing is bad for low types, who reveal that they are weak competitors, but is advantageous for high types, because they are able to signal their strength. We identify a so-called "briber's curse," whereby acceptance of a proposal causes the proposer to downgrade her expected value for the object. When there is additional competition in the form of bidders outside of the cartel, the briber's curse significantly harms cartel profits and raises seller revenues.

JEL classification: D44; D82

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1. Introduction

Collusion is a well-documented phenomenon in auctions, one of first-order interest to sellers.[2] In one-shot games, collusion agreements are typically analyzed using standard tools of mechanism design, invoking an uninformed third party (UTP) that designs and enforces the mechanism. Seminal papers in this regard are Graham and Marshall (1987) and McAfee and McMillan (1992), who find optimal collusive mechanisms in independent private values models in second price and first price auctions, respectively (see also Mailath and Zemsky, 1991 and Marshall and Marx, 2007). A key feature of this approach is that commitment is often assumed ex-ante, whereas in many cases, bidders may have private information at the time that they commit to the ring. For example, Hendricks et al. (2008) study auctions for oil and gas leases during a period over which pooling of information and joint bidding was legal and contractible (so the enforcement problem is not an issue), and yet still find little incidence of such behavior. They build a model in which bidders observe a collusive mechanism designed by an outsider but commit at the interim rather than the ex-ante stage, therefore leading bidders with high signals to forego joining the ring.[3]

Under the UTP approach, the goal of the designer of the collusive mechanism is to maximize the joint surplus of the ring; however, when the designer is one of the bidders, her objective is to maximize her own individual surplus. Further, when the bidders have private information before putting forth a proposal to collude, they must be cognizant of the signaling aspects that the mere act of proposing carries. Through such a proposal, a bidder will necessarily leak information about her signal to others, which can negate some of the benefits of colluding in the first place.

To analyze these issues in a general setting, we build on the model of Eső and Schummer (2004) (hereafter, ES), who study collusion in a second-price auction (SPA) with two bidders with independent private values (IPV). After receiving their signals, bidder 1 approaches bidder 2 with a collusion proposal, which consists of bidder 2 sharing her information in exchange for a transfer and the commitment to staying out of the auction.[4] If the agreement is rejected, bidders compete non-cooperatively in the auction using beliefs that are updated with any information revealed in the first stage. In the IPV setting, ES show that there is a unique continuous equilibrium in which collusion occurs, and that collusion is (socially) inefficient.

We allow for general interdependent valuations and affiliated signals. When valuations are interdependent, going beyond the uninformed third party approach becomes even more relevant than with private values. Indeed, in their seminal paper, McAfee and McMillan (1992) say that

[2] For example, Porter and Douglas Zona (1993) study collusion in highway procurement auctions, and Baldwin et al. (1997) study collusion in forest timber auctions.

[3] Hendricks et al. (2008) actually study two different models that differ in their assumptions about beliefs. In one case, beliefs are allowed to respond (or update) according to information revealed before the auction, while in the other, beliefs are "passive" and do not change. In common value settings, they find collusion to be inefficient (from the ring's perspective) only under passive beliefs; when beliefs are allowed to adapt, efficient collusion remains possible. This is in contrast to our results, where collusion will be inefficient in a model that allows for updating of beliefs.

[4] Information leakage and commitment are both important issues in modeling collusive agreements. Our goal here is to highlight the former, and so, for simplicity, we have abstracted away from commitment issues. In general, this could be solved by embedding our stage game in a repeated game, and using continuation values to sustain commitment. Note, however, that the information leakage problem would still be present.

they focus on private values because "in the pure common-value case, the optimal cartel mechanism is simple", and the cartel will be able to extract all of the surplus from the seller (i.e., collusion is not only socially efficient, but is also efficient from the perspective of the ring). When there is no such outsider to design and enforce a collusive mechanism, this will no longer be true. Further, the incentives to collude are even stronger with interdependent values, because of an information sharing motive that is not present in the IPV setting: colluding bidders can pool their information to refine their value estimate. This aids in determining whether the object is more valuable than any potential reserve price and in optimizing bids against bidders outside of the cartel.

We begin by characterizing the unique continuous, partially separating equilibrium of this game, which is a generalization of the equilibrium identified by ES under private values. We then move to analyze overall efficiency and the distribution of surplus between bidder 1 (the proposer), bidder 2 (the receiver), and the seller, and compare to both a competitive auction with no collusion and the standard UTP approach to collusion. With regard to efficiency, when the setting is one of pure common values, the equilibrium is trivially socially efficient, but, from the perspective of the ring, it will not be collusively efficient, in the sense that the seller will retain some profit. If there is any private value component to valuations (as in ES, where values are completely private), the equilibrium will not even be socially efficient, because bidder 2 may take the transfer rather than participate in the auction, even though she has a higher valuation for the object.

We next consider the distribution of surplus within the cartel itself. Recall that under the uninformed third party approach, all bidders are symmetric, which is no longer true in our model. Even though the bidders are no longer symmetric and the cartel will leave some surplus to the seller, both bidders are still better off, type by type, compared to a competitive auction. While this is perhaps not surprising, more interesting is the distribution of surplus between the (asymmetric) bidders. While the act of proposing leaks information to the receiver, it also allows her to show her "strength". From the interim perspective, we show that in general, there is no dominance relation between the payoffs of the proposer and those of the receiver: low types prefer to receive, while high types prefer to propose. From the ex-ante perspective, we continue to find no payoff dominance between the proposer and the receiver, with the key factor here being the degree of correlation between the bidders' information: when the correlation is low, the mass of types who would prefer to be the proposer is also low, and so ex-ante, the receiver is relatively better off; however, when correlation is high, bidder 1's information is already relatively informative of bidder 2's information, and so the mass of types who prefer to be the proposer is high, and correspondingly, from the ex-ante perspective, the proposer is relatively better off.[5]

Last, most of the literature on collusion considers all-inclusive cartels (with a prominent exception being Marshall and Marx, 2007, which is discussed below). However, it is also important to understand what happens when cartels are not all-inclusive. This becomes particularly relevant in the interdependent value setting we consider, because interdependence provides an additional motivation for colluding (beyond decreasing competition): cartel members can pool information, which can help to optimize bids against outside competition. We find that ex-ante payoffs are highly sensitive to the introduction of an outside bidder. While we continue to find benefits to collusion, we also find that the magnitude of the gains is greatly decreased relative to the case of

[5] We say "relatively better off" because both bidders are better off compared to no collusion, but one bidder will gain more than the other.

an all-inclusive ring, bidder 1's offers to bidder 2 are significantly lower, and colluding provides only a marginally greater expected payoff than not. This can be interpreted as a type of "winner's curse" (perhaps more accurately labeled a "briber's curse"): the fact that her proposal is accepted causes bidder 1 to revise her expectation for the object's value downwards.[6] Thus, in equilibrium, bidder 1's offer to bidder 2 is significantly lower in the presence of outside competition, which in turn causes bidder 2 to accept less often, which results in a more competitive final auction. This leads to significantly higher revenue for the seller.

As alluded to above, Marshall and Marx (2007) allow for outside bidders in an IPV framework using the UTP approach to modeling the collusive mechanism, and find that there is some leakage of surplus to outside bidders with first-price auctions, but not with second-price auctions.[7] The reason is that, in their model, the ring will send the highest-value bidder to the auction, and it is a dominant strategy at the auction stage for all bidders to bid their true valuations. Since this is the same as what she would have done in the absence of collusion, for outside bidders, the outcome is equivalent.[8] In our model, this need no longer be true, since it is not necessarily the cartel member with the highest value that is sent to the auction. In fact, under common values, we show that not only can collusion between bidders 1 and 2 make bidder 3 (the outsider) better off compared to no collusion, bidder 3 can actually gain *more* than bidder 1, who is a member of the cartel. The intuition is related to the "briber's curse" alluded to above: at the auction stage, the highest competing bid is revised downward relative to a competitive auction, which helps the outside bidder. For completeness, we also analyze the introduction of an outside bidder under private values. In contrast with Marshall and Marx (2007), we find that the outsider can again be made strictly better off. The reason is that, the maximum bid submitted by bidders 1 and 2 is either (i) the same as it would have been without collusion or (ii) strictly lower. The latter case happens when a relatively low type of bidder 1 bribes out a higher type of bidder 2, and then goes on to lose to bidder 3, who now pays less.[9] Because it is not necessarily the highest-value cartel bidder that is sent to participate in the auction, the outside bidder can free-ride on bidder 1's payment to bidder 2 to stay out of the auction and be made strictly better off.

1.1. Related literature

As mentioned in the introduction, our model builds on that of ES, who study collusion and signaling in a setting of independent private values (IPV). We generalize their results in several ways. First, many auctions do not fit the IPV framework. Our setting allows for both general interdependent valuations and correlated signals. This complicates the analysis, but allows us to analyze settings where the incentive to collude may actually be stronger than in the IPV framework, due to the aforementioned information-sharing motive when values are interdependent. Second, we provide new results and insights with regard to overall efficiency and the distribution of surplus that have no analogue in ES. In particular, we provide answers to the question of whether a bidder prefers to be the proposer or the receiver. While information leakage harms

[6] While this issue is also present even with an all-inclusive cartel, the presence of an outsider compounds it, because not only must bidder 1 pay bidder 2, she must then still go to the auction and compete against a third bidder.

[7] Seres (2015) extends the framework of Marshall and Marx (2007) to include interdependent values, and shows that incentive-compatible bid coordination mechanisms exist only under independent private values.

[8] Of course, the cartel bidders are better off, because by suppressing the bids of the other cartel members, the eventual winner may pay a lower price (when the first and second highest values are both bidders inside the cartel).

[9] I thank an anonymous referee for pointing out this simple intuition.

low types, who are worse off being the proposer, the situation is reversed for high types, who are better off being the proposer because they can signal their strength. This ambiguity persists from the ex-ante perspective, depending on how highly correlated the bidders' information is. Third, much of the work on collusion considers only all-inclusive cartels. We extend our analysis to include a bidder from outside of the cartel, in both interdependent value and IPV settings. We find that this can have a disproportionately large effect on the payoffs to all players, including the seller, who receives significantly higher profits. We view these latter two results in particular as a contrast to the UTP approach to collusion, and they highlight potential difficulties that are not present when there is a disinterested mediator.[10]

Other recent papers that consider closely related models of collusion are Rachmilevitch (2013), who studies first-price auctions, Rachmilevitch (2015), who adds a second round in which bidder 2 is able to counter-propose, and Balzer (2015), who shows that efficient collusion is not possible even when the principal is allowed to propose a more general set of mechanisms than simple transfers. All of these papers work in the IPV framework and consider only 2 bidders, while we allow for interdependent values and correlated signals, and consider bidders outside of the cartel. Chen and Tauman (2006) consider a similar IPV model where the collusive bidders' valuations can take on two or three types and, after accepting, the receiver can re-enter the auction using a shill bidder. Kivetz and Tauman (2010) study a bribing game in first-price auctions assuming complete information among the bidders. Recent experimental papers by Llorente-Saguer and Zultan (2014) and Agranov and Yariv (2015) study first and second price auctions allowing for pre-auction communication and collusion in the lab, again in IPV settings. Hendricks et al. (2008) allow for interdependent values, but take the UTP approach to designing the collusive mechanism and information leakage comes only from the yes/no decision to participate in the collusive mechanism or not. Another strand of the literature asks the question of whether it is possible to design an auction that is "collusion-proof". Papers in this regard include Dequiedt (2007), Pavlov (2008), and Che and Kim (2009). These papers again consider the case of independent private values, and use the UTP approach when modeling collusive behavior. Given this model of collusive behavior, they then look for an optimal design response from the seller. Whether allowing for interdependent values and signaling amongst the collusive bidders makes collusion-proofness easier or harder to achieve is an interesting open question for future work.

Beyond auctions, problems of mechanism design by privately involved parties are prevalent in economics, from bilateral or multilateral trade with an informed seller, to contracting with privately informed firms. The literature stems from the seminal work of Myerson (1983) and Maskin and Tirole (1990, 1992). More recent work includes Mylovanov and Troeger (2012) and Mylovanov and Troeger (2014). These recent papers characterize the problem of mechanism design by an informed principal, whose information is payoff-irrelevant to the agents (but is relevant in determining her behavior in the mechanism). Quesada (2005) introduces the informed-principal approach to a problem of collusion in production delegation.

The remainder of the paper is organized as follows. Section 2 sets up the basic model by introducing the payoffs, information structure, and timing of the game. Section 3 analyzes the equilibrium strategies, while Section 4 studies efficiency and the distribution of surplus among the (asymmetric) bidders and the seller. Section 5 introduces competition outside of the cartel.

[10] We do not mean to suggest that collusion is unimportant or should be ignored; indeed, collusion is very likely to occur in equilibrium in the model we present. At the same time, we think that our model provides additional insights that complement other approaches to modeling collusion and hope they aid in its detection and the formulation of appropriate responses.

Section 6 concludes. Proofs of any results not in the main body of the paper can be found in the appendix.

2. Model

There are two bidders, as well as a seller. Bidder 1 is the "informed principal," endowed with the bargaining power to approach bidder 2. The auction is a sealed-bid second-price auction (SPA). Before any social interaction, each bidder receives a private signal S_i, and in addition, there is another (unobservable) state of the world Z which may influence the value of the object but is not seen by either bidder. Let $F(z, s_1, s_2)$ be the joint cumulative distribution function of (Z, S_1, S_2). We assume that F is symmetric in (s_1, s_2), and that all of the variables are affiliated (the extreme case of independence is allowed). In addition, we normalize all signals to lie in the interval $[0, 1]$, and assume that $F(z, s_1, s_2)$ has a continuous, strictly positive density function over this region, denoted $f(z, s_1, s_2)$.

Each bidder has a valuation function $V_i = V(Z, S_i, S_{-i})$, where V is nonnegative, continuous, and nondecreasing in all variables. We assume that all necessary expectations exist and are finite. This formulation encompasses both the independent private-values and pure common-values models, as well as a range of intermediate models.[11]

The timing of the game is as follows.

Stage 0: Nature draws z, s_1 and s_2; bidder i privately observes s_i.

Stage 1 (Collusion stage): Bidder 1 has the option of offering a collusion contract to bidder 2, which consists of a transfer t paid to bidder 2 in exchange for a report of her (verifiable) signal s_2. Upon receiving a proposal, bidder 2 either accepts or rejects the contract. If she accepts, she reports s_2, receives transfer t, and does not participate in the auction.

Stage 2 (Auction stage): If bidder 2 accepts the contract, she commits to staying out of the auction. If bidder 2 rejects the contract, both agents bid non-cooperatively.

The payoff to bidder 1 with value V_1 who faces a payment of m_1 to the auctioneer and who pays bidder 2 a transfer of t is $\mathbb{I}(1 \text{ wins object})V_1 - m_1 - t$. Bidder 2's payoff, when his value is V_2, pays m_2 to the auctioneer and receives t from bidder 1 is $\mathbb{I}(2 \text{ wins object})V_2 - m_2 + t$.

3. Equilibrium

3.1. Auction stage

We begin by solving the auction stage in the case that collusion does not occur. Define the function $\omega(x, y) = E[V_i | S_i = x, S_{-i} = y]$. In words, this is bidder i's expected value for the object, conditional on her own signal being x and that of her opponent being y. Milgrom and Weber (1982) show that $\omega(x, y)$ is nondecreasing in both arguments; we make the non-degeneracy assumption that this function is strictly increasing in x, is supermodular in (x, y), and $\omega_1 \geq \omega_2$.[12] Last, we normalize $\omega(0, 0) = 0$.

[11] The payoff structure is very similar to Milgrom and Weber (1982).

[12] The subscripts denote partial derivatives. These are all fairly natural assumptions on how a bidder's valuation depends on his own information relative to that of his opponent (see, for example, Dasgupta and Maskin, 2000 and

Proposition 1 *(Milgrom and Weber, 1982). In any continuation game following failure to collude, the bidding profile $\beta(s_i) = \omega(s_i, s_i)$ for $i = 1, 2$ is an ex-post equilibrium.*

This is the standard equilibrium selection for second price auctions used in the literature, because it is symmetric and belief-free (i.e., it is an ex-post equilibrium), and it is the one we will select for our analysis. In the case of independent private values (which we will discuss in Section 5), it reduces to the familiar dominant strategy equilibrium where each bidder simply bids her valuation.[13]

3.2. Collusion stage

We can now move back to the collusion proposal/acceptance stage of the game. A strategy for bidder 1 in the first stage can be identified by a function mapping her signal into transfers (or "bribes"). We will denote this function $\bar{t} : [0, 1] \to \mathbb{R}_+$, and write $\bar{t}(s_1)$ for the transfer that type s_1 offers to bidder 2.[14] A strategy for bidder 2 in the first stage of the game can be identified by an *acceptance correspondence* $\mathcal{A} : \mathbb{R}_+ \to 2^{[0,1]}$, where $\mathcal{A}(t)$ is the (measurable) set of types that accept transfer t, while the types in the complement reject it.

We can write bidder 1's payoff if she is of type s_1, offers transfer t, has beliefs $\tilde{F}_2(\cdot)$ over s_2, and bidder 2's acceptance set is A as

$$U^1(s_1, t; A, \tilde{F}_2) = \int \{\mathbb{I}(s_2 \in A)[\omega(s_1, s_2) - t] + \mathbb{I}(s_2 \notin A \text{ and } s_2 < s_1)[\omega(s_1, s_2) - \omega(s_2, s_2)\} d\tilde{F}_2(s_2) \tag{1}$$

Similarly, we can write bidder 2's payoff if she is of type s_2, has beliefs over s_1 represented by $\tilde{F}_1(\cdot)$,[15] and rejects the contract, as:

$$U^2(s_2; \tilde{F}_1) = \int [\omega(s_2, s_1) - \omega(s_1, s_1)] d\tilde{F}_1(s_1).$$

Krishna, 2003). They will hold in a wide range of commonly used models, including pure private values, pure common values, or a combination of a common value component plus an additional private value component, as in Bulow and Klemperer (2002) (e.g., $V(Z, S_i, S_j) = (1 + \alpha)S_i + S_j$), among others.

[13] This is not the only equilibrium of a second-price auction. In fact, with two bidders and pure common values ($V_i = Z$ for $i = 1, 2$), Milgrom (1981) shows that there is a family of ex-post equilibria, indexed by a strictly increasing continuous real function h, in which bidders 1 and 2 play the following strategies: $\beta_1(s_1) = E[Z|S_1 = s_1, S_2 = h(s_1)]$ and $\beta_2(s_2) = E[Z|S_1 = h^{-1}(s_2), S_2 = s_2]$. We focus on the symmetric equilibrium where h is the identity function. This is the equilibrium commonly used in the auction literature, and, with 3 or more bidders, Bikhchandani and Riley (1991) show that the symmetric equilibrium is the only equilibrium in strictly increasing and continuous strategies in which all bidders have a positive ex-ante probability of winning. Therefore, comparability of the results with those of the case of 3 bidders (which we will study in Section 5) is another rationale for focusing on this equilibrium.

[14] While we only allow bidder 1 to offer a payment to bidder 2, this is without loss of generality: if bidder 1 were given the option to demand a payment from bidder 2 in exchange for committing to stay out of the auction herself (which can be modeled as allowing $\bar{t}(\cdot)$ to be negative), she would never actually make use of it in equilibrium. The reason is that offering a transfer (either positive or negative) allows high types of bidder 1 to reveal their strength, but in order to make this credible and prevent lower types from deviating, the signal must also be sufficiently "painful". When $\bar{t}(\cdot)$ is positive, these two opposing effects can be balanced; if $\bar{t}(\cdot)$ were negative, both effects would go in the same direction, and so it would not be possible for strong types of bidder 1 to signal this in a way that is credible. I thank an anonymous referee for raising this question.

[15] These beliefs may be updated beliefs (for example, using information about s_1 that is revealed through the observed transfer t).

With the fixed continuation equilibrium in the noncooperative auction, we will refer to $(\bar{t}(\cdot), \mathcal{A}(\cdot), \tilde{F}_1, \tilde{F}_2)$ as a **Bayesian equilibrium** if[16]:

(i) For all s_1, $\bar{t}(s_1) \in \arg\max_t U^1(s_1, t; \mathcal{A}(t), \tilde{F}_2)$.
(ii) For all t, $s_2 \in \mathcal{A}(t)$ if and only if $t \geq U^2(s_2; \tilde{F}_1)$.
(iii) The beliefs $\tilde{F}_1$ and $\tilde{F}_2$ follow Bayes' rule, where possible.

3.3. Existence

As is common in signaling games, there may be many equilibria. For example, in the IPV setting, ES show that "no collusion" ($\bar{t}(s_1) = 0$ for all s_1 and bidder 2 rejects every possible offer) is an equilibrium of this game, which then just reduces to a standard second price auction. Because all offers other than 0 are off the equilibrium path, beliefs for bidder 2 are unrestricted, and this equilibrium can be supported by bidder 2 having the most optimistic beliefs possible that assign probability 1 to bidder 1 being of type of type $s_1 = 0$ for any out-of-equilibrium t. In our model with interdependent values, there will be additional equilibria that are qualitatively similar, where $\bar{t}(s_1) = t^*$ for all s_1 (i.e., complete pooling), provided that the constant t^* is not too high.[17] However, these equilibria rely on delicate out-of-equilibrium beliefs and can be ruled out using standard forward induction refinements. Following ES, we look for an equilibrium that is continuous and partially separating, and therefore robust to out-of-equilibrium beliefs. We first prove the existence of such an equilibrium, and then discuss uniqueness.

In a partially separating equilibrium, bidder 1's strategy is continuous and strictly increasing in his type, up to a critical point s_1^* at which all types $s_1 \geq s_1^*$ pool at some fixed t^* (see Fig. 1 for an illustration). In this equilibrium, stronger types of bidder 1 signal their strength by offering more to bidder 2 to get her to stay out of the auction, up to the point t^* at which all types of bidder 2 agree to stay out. We can use local incentive compatibility (IC) constraints to characterize $\bar{t}$. Assume that bidder 2's strategy takes the form of a *cutoff strategy* $\mathcal{A}(t) = [0, \bar{s}_2(t)]$; that is, upon observing an offer of t, there is a cutoff type $\bar{s}_2(t)$ such that all types $s_2 \leq \bar{s}_2(t)$ accept, and all types $s_2 > \bar{s}_2(t)$ reject t. We first (informally) derive the local IC constraints in the separating region, and then prove that these strategies do indeed constitute an equilibrium (Theorem 1). In particular, if bidder 1's strategy is summarized by $\bar{t}(s_1)$, then her expected utility when she is of true type s_1 in the separating region and mimics a nearby type $\hat{s}_1$ is:

$$\int_0^{h(\hat{s}_1, \bar{t}(\hat{s}_1))} [\omega(s_1, s_2) - \bar{t}(\hat{s}_1)] dF(s_2|s_1), \tag{2}$$

where we have explicitly written $h(\hat{s}_1, \bar{t}(\hat{s}_1))$ as the type of bidder 2 who is indifferent to accepting $\bar{t}(\hat{s}_1)$, conditional on believing that bidder 1's type is $\hat{s}_1$ with probability 1.[18] Local IC

[16] As in many games with a continuum of types, boundary types may be indifferent between two actions. We will ignore such issues, as the decisions of such boundary types are effectively irrelevant, and two equilibria that differ only in the behavior of boundary types are essentially equivalent.

[17] Values $t^* > 0$ are not sustainable under private values, because there will always be a mass of types with sufficiently low valuations $s_1 < t^*$ such that paying t^* leads to a strictly negative payoff. With interdependent values, these types may be willing to offer $t^* > 0$ in equilibrium, as long as they can pool with higher types and t^* is low enough that their overall payoff is positive in expectation (integrating over bidder 2's signal).

[18] In the separating region, $\bar{t}(\cdot)$ will be invertible, and so bidder 2 perfectly observes bidder 1's type.

requires that this function be maximized at $\hat{s}_1 = s_1$. For now assuming the requisite differentiability, we can solve the first-order condition to get an equation for $\bar{t}'(s_1)$ (the notation $h_i(\cdot,\cdot)$ indicates the partial derivative of h with respect to the i^{th} argument):

$$t'(s_1) = \frac{[\omega(s_1, h(s_1, \bar{t}(s_1))) - \bar{t}(s_1)] f(h(s_1, \bar{t}(s_1))|s_1) h_1(s_1, \bar{t}(s_1))}{F(h(s_1, \bar{t}(s_1))|s_1) - [\omega(s_1, h(s_1, \bar{t}(s_1))) - \bar{t}(s_1)] f(h(s_1, \bar{t}(s_1))|s_1) h_2(s_1, \bar{t}(s_1))} \tag{3}$$

In the above equation, we need the function h, which is defined implicitly as follows:

$$t = \omega(h(s_1, t), s_1) - \omega(s_1, s_1). \tag{4}$$

That is, $h(s_1, t)$ is the type of bidder 2 who is indifferent between accepting and rejecting t conditional on knowing s_1.[19] Using the implicit function theorem, we find:

$$h_1(s_1, t) = \frac{\omega_1(s_1, s_1) + \omega_2(s_1, s_1) - \omega_2(h(s_1, t), s_1)}{\omega_1(h(s_1, t), s_1)} \quad \text{and} \quad h_2(s_1, t) = \frac{1}{\omega_1(h(s_1, t), s_1)}.$$

Note that $h(s_1, t)$ is increasing in both arguments. Substituting into (3), we find that in the separating region we have

$$\bar{t}'(s_1) = \frac{[\omega(s_1, h(s_1, \bar{t}(s_1))) - \bar{t}(s_1)][\omega_1(s_1, s_1) + \omega_2(s_1, s_1) - \omega_2(h(s_1, \bar{t}(s_1)), s_1)] f(h(s_1, \bar{t}(s_1))|s_1)}{\omega_1(h(s_1, \bar{t}(s_1)), s_1) F(h(s_1, \bar{t}(s_1))|s_1) - [\omega(s_1, h(s_1, \bar{t}(s_1))) - \bar{t}(s_1)] f(h(s_1, \bar{t}(s_1))|s_1)}.$$

The above derivation suggests the following result. The full proof of equilibrium existence can be found in the appendix.

Theorem 1. *Let $h(s_1, t)$ solve $t = \omega(h(s_1, t), s_1) - \omega(s_1, s_1)$, and assume the following regularity conditions hold: (i) $\omega_{11}(x, y) \geq 0$ and (ii) the conditional CDF $F(s_2|s_1)$ is log-concave in s_2. Then, the following differential equation with initial condition $\bar{t}(0) = 0$ has a unique, continuous solution:*

$$\bar{t}'(s_1) = \begin{cases} \frac{[\omega(s_1, h(s_1, \bar{t}(s_1))) - \bar{t}(s_1)][\omega_1(s_1, s_1) + \omega_2(s_1, s_1) - \omega_2(h(s_1, \bar{t}(s_1)), s_1)] f(h(s_1, \bar{t}(s_1))|s_1)}{\omega_1(h(s_1, \bar{t}(s_1)), s_1) F(h(s_1, \bar{t}(s_1))|s_1) - [\omega(s_1, h(s_1, \bar{t}(s_1))) - \bar{t}(s_1)] f(h(s_1, \bar{t}(s_1))|s_1)}, \\ \quad h(s_1, \bar{t}(s_1)) < 1 \\ 0, \quad \text{otherwise} \end{cases} \tag{5}$$

Further, let s_1^* be such that $h(s_1^*, \bar{t}(s_1^*)) = 1$, and $t^* = \bar{t}(s_1^*)$, and define

$$\bar{s}_2(t) = \begin{cases} h(\bar{t}^{-1}(t), t), & t < t^* \\ 1, & t \geq t^* \end{cases}.$$

Then, there exists an equilibrium in which bidder 1 uses strategy $\bar{t}(s_1)$ and bidder 2 uses strategy $\mathcal{A}(t) = [0, \bar{s}_2(t)]$.

It is not hard to show the cutoff strategy given is optimal for bidder 2 when she perfectly observes bidder 1's type, and, for the remainder of the paper, we will simply identify bidder 2's strategy with the cutoff function $\bar{s}_2(t)$ (and drop the $\mathcal{A}(t)$ notation). The more difficult part is to show that $\bar{t}(s_1)$ is indeed optimal for bidder 1. The interdependence of the signals introduces

[19] Because ω is strictly increasing in its first argument, this equation has a unique solution $h(s_1, t)$ for every (s_1, t), and $h(s_1, t) > s_1$ for all $t > 0$.

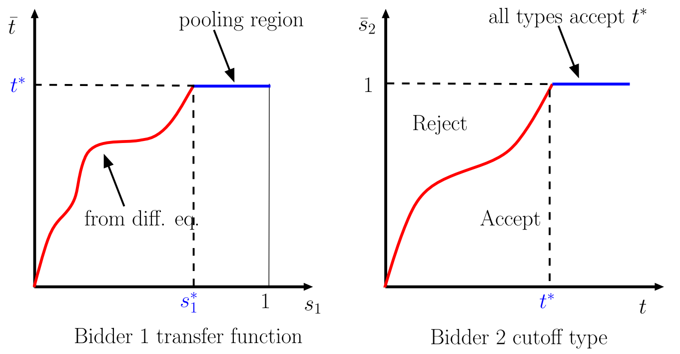

Fig. 1. The continuous, partially separating equilibrium.

nontrivial complications, since a small change in bidder 1's type s_1 affects not only her direct payoff, but also her beliefs about the distribution of signals observed by bidder 2. This means that, for any fixed t, a change in bidder 1's own signal s_1 affects the probability of t being accepted, which introduces complications not present in the simpler IPV setting. To solve this, we rely on the theory of monotone comparative statics under uncertainty developed by Athey (2002) (see also Athey, 1996), in which affiliation (which is equivalent to log-supermodularity of the joint pdf) plays a key role. We show that bidder 1's payoff function under certainty (that is, if she offered t conditional on knowing both s_1 and s_2), $v(s_1, s_2, t)$, satisfies a relevant single-crossing property in (s_2, t). When the signals are affiliated, this is sufficient for bidder 1's expected payoff (integrating over s_2, conditional on s_1), to satisfy single-crossing in (s_1, t). Once we know this, standard envelope theorem arguments deliver that $\bar{t}(s_1)$ is optimal.

An illustration of these strategies is given in Fig. 1. The regularity conditions are sufficient conditions to ensure that equation (5) admits a continuous solution.[20] Note that when bidder 1 is the strongest type, bidder 2 knows that she will lose for sure in a competitive auction and receive a payoff of 0, and so she is willing to accept any small transfer. However, in order to be credible, the transfer must be high enough to keep lower types from deviating. The differential equation characterizes exactly how this is done, up to the point t^* where bidder 1 is strong enough such that if bidder 2 knew bidder 1's type, she would accept with certainty. Since bidder 1 has proved her strength, there is no reason to offer anything above t^*, and so the remaining types pool at t^*.

3.4. Uniqueness

Theorem 1 tells us that a continuous, partially separating equilibrium in which stronger types of bidder 1 offer strictly more up to the point where all types of bidder 2 accept exists.[21] What about other equilibria? As discussed in the previous section, it is obvious that $\bar{t}(s_1) = 0$ and bidder 2 always rejecting is an equilibrium in which no collusion ever occurs and payoffs are

[20] Log-concavity is a standard assumption widely-used in the mechanism design literature. We require log-concavity of the conditional CDF $F(s_2|s_1)$ in s_2. When the signals are independent, this reduces to log-concavity of the CDF $F(s_2)$, which is weaker than the (widely-used) assumption that the corresponding density $f(s_2)$ is log-concave (Bagnoli and Bergstrom, 2005 provide a comprehensive discussion of log-concavity and its applications in economics). More generally, log-concavity of $F(s_2|s_1)$ in s_2 is implied by the (stronger) assumption that the joint density function $f(s_1, s_2)$ be log-concave (see Prékopa, 1980). ES also assume log-concavity of the (in their case independent) distribution functions.

[21] I thank an anonymous referee for suggesting the analysis in this section.

equivalent to a standard competitive auction. Under IPV, this will be the only other continuous equilibrium (besides the equilibrium of Theorem 1, see ES). With interdependent values, there are more pooling equilibria, described by $\bar{t}(s_1) = t^*$ for all s_1 for some constant t^* that is not too high.[22] Say that an equilibrium is *completely pooling* if $\bar{t}(s_1) = t^*$ for all s_1; otherwise, it is *(partially) separating*. Note that completely pooling equilibria require very delicate out-of-equilibrium beliefs and will not survive the standard forward induction refinements commonly used in signaling games. The equilibrium identified in Theorem 1, on the other hand, is robust to out-of-equilibrium beliefs.[23] Further, we can say the following.

Theorem 2. *The equilibrium identified in Theorem 1 is the unique continuous, partially separating equilibrium.*

The proof in the appendix shows two things: first, if $\bar{t}(0) > 0$, then the function must be pooling at $\bar{t}(0)$ on some non-degenerate domain. If this were not true, then there is separation at 0 and low types of bidder 1 reveal that they are weak. Thus, they will only be willing to offer a small amount, and, in particular, any amount they would be willing to offer would be rejected by bidder 2. Second, we show that if there is pooling to the left of some $\hat{s}_1$, then there cannot be separation to the right of $\hat{s}_1$. If there were, type $\hat{s}_1$ would mimic a slightly higher type; while this requires making a greater offer, it is more than offset by the fact that it separates him from the lower types in the pooling region. Thus, equation (5) characterizes the unique partially separating equilibrium. Further, it is easy to see that full separation is impossible, as the strongest types of bidder 1 will always pool: if bidder 1 reveals herself as sufficiently strong (i.e., $s_1 \in (s_1^*, 1]$ for some $s_1^* < 1$) bidder 2 will (strictly) prefer to accept with probability 1, and thus sufficiently high types should mimic lower types. By doing so, they will induce the same action from bidder 2 while paying her less money.

3.5. Example: pure common values

The differential equation (5) is complex, and generally will need to be solved numerically. However, if the structure of the valuation and signal distribution functions are simple enough, closed-form solutions are possible, and can help illustrate the general ideas more clearly. Consider a case of pure common values, $V(Z, S, S_j) = \frac{S_i + S_j}{2}$ for $i, j = 1, 2$. The signals S_1 and S_2 are distributed independently and uniformly on the interval [0, 1] (the random variable Z is irrelevant for this example). This implies that $\omega(x, y) = \frac{x+y}{2}$.

This example satisfies all of conditions needed for Theorem 1. Thus the differential equation (5) has a unique, continuous solution $\bar{t}$, and further, it is possible to solve for $\bar{t}(\cdot)$ analytically. In the separating region, equation (5) becomes

$$\bar{t}'(s_1) = \begin{cases} \frac{s_1}{2t(s_1) - s_1}, & s_1 + 2t(s_1) < 1 \\ 0, & \text{else} \end{cases}.$$

The solution to this equation is $\bar{t}(s_1) = \min\{s_1, 1/3\}$. Similarly, if bidder 2 perfectly observes s_1 from the offered transfer, the cutoff type will be $h(s_1, t) = 2t + s_1$. Thus, the equilibrium strategy for bidder 2 is summarized by $\bar{s}_2(t) = \min\{3t, 1\}$. These strategies are shown in Fig. 2.

22 See footnote 17.

23 Note that in this equilibrium, bidder 1 offering an out-of-equilibrium transfer (above t^*) is a dominated strategy.

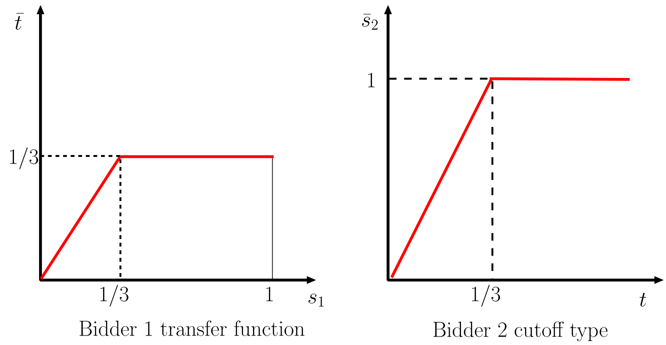

Fig. 2. Equilibrium strategies for the common value example.

4. Outcomes

4.1. Equilibrium and surplus distribution

Now that we have characterized the unique robust, continuous equilibrium, we can use these results to ask important questions relating to both efficiency and the successfulness of collusion. We say that an equilibrium is **efficient** if the good is won by the bidder with the higher valuation for all realized (s_1, s_2). Recall that the case of pure common values is a special case of our model, in which $V(z, s_i, s_j) = V(z, s_j, s_i)$ for all s_i, s_j. In this case, the outcome is trivially efficient no matter who gets the good. However, if there is any private value component to the valuations, there will be inefficiency (as noted by ES in the case of pure private values). This can be easily seen by defining $s^* = \inf_{s_1}\{\bar{t}(s_1) = t^*\}$, i.e., s^* is the lowest type of bidder 1 that pools at t^*. All types of bidder 2 accept t^* when offered and stay out of the auction, and so for all types of bidder 1 in the pooling region, they win the auction with certainty. Thus, any realization (s_1, s_2) such that $s_2 > s_1 \geq s^*$ is inefficient, because bidder 2 will accept and bidder 1 will get the object, even though she values it less than bidder 2. For completeness, we state this result as the following corollary.

Corollary 3. *If, for all $s_i > s_j$, we have $V(z, s_i, s_j) > V(z, s_j, s_i)$, then the equilibrium is inefficient.*

There is an another notion of efficiency from the cartel's perspective that relates to whether or not the cartel is able to capture all of the surplus from the seller. According to this definition, there is inefficiency even under pure common values, because sometimes bidder 2 will reject (when s_2 is high and s_1 is low), and the seller will thus receive some revenue. This is contrast to the uninformed third party approach to modeling collusion under common values, where an all-inclusive cartel is able to design an optimal collusive mechanism that extracts all surplus from the seller.

Since collusion is a way for a group of bidders acting jointly to transfer rents from the seller to themselves, it seems likely that the loss in surplus will at least partially be borne by the seller, who will be worse off than if the bidders bid competitively. But, because in our model the bidders are asymmetric, what is less obvious is how the surplus is distributed among the bidders themselves, and how they fare compared to a competitive auction. The next two theorems speak to these questions, starting with the latter. First, define the interim expected utility functions of the bidders (conditional on their signals) as:

$$U^1_{Int}(s_1) = \begin{cases} \int_0^{\bar{s}_2(\bar{t}(s_1))} [\omega(s_1, s_2) - \bar{t}(s_1)] dF(s_2|s_1), & s_1 \le s_1^* \\ \int_0^1 [\omega(s_1, s_2) - t^*] dF(s_2|s_1), & s_1 > s_1^* \end{cases} \tag{6}$$

$$U^2_{Int}(s_2) = \int_0^{\hat{s}_1(s_2)} [\omega(s_2, s_1) - \omega(s_1, s_1)] dF(s_1|s_2) + \int_{\hat{s}_1(s_2)}^1 \bar{t}(s_1) dF(s_1|s_2) \tag{7}$$

where s_1^* is the lowest type of bidder 1 who pools at transfer t^* in equilibrium, and $\hat{s}_1(s_2)$ is the lowest type of bidder 1 who, in equilibrium, induces bidder 2 to accept her equilibrium transfer when of type s_2 (defined implicitly as $\hat{s}_1(s_2) = \min\{s_1 : \bar{s}_2(\bar{t}(s_1)) = s_2\}$).[24]

Theorem 4. *Both bidders are better off, type by type, in the game with collusion than in a standard competitive second price auction with no ex-ante collusion stage. Correspondingly, the expected revenue received by the seller is less with collusion.*

Theorem 4 shows that every type of every bidder is better off with collusion than without, and the seller is worse off. This is easy to see for bidder 1, because he always has the option of deviating to not offering any collusion proposal (which we model as offering $t = 0$), which induces bidder 2 to always reject, thereby giving bidder 1 the same (expected) payoff he would have in a standard second price auction. For bidder 2 of type s_2, if her type is very low, then she would have lost the auction anyway, and so is better off accepting. If her type is sufficiently high that she rejects, she wins and the outcome is equivalent to a competitive auction. On the intermediate range, she would have won a competitive auction, but her gains net of what she must pay the seller are less than the transfer she gets in the collusive model.

Since our model is asymmetric, a more interesting question is how the rents extracted by the ring are distributed between its members. Bidder 1 necessarily reveals some information when he makes an offer to bidder 2 which can then be used by bidder 2 to improve her payoff. On the other hand, the opportunity to reveal information may also be advantageous to bidder 1, because it allows her to signal her "strength" and convince bidder 2 to stay out for less than it would cost her to win a competitive auction. The question then becomes which of these effects is stronger. Intuitively, for low types, the former effect dominates, while as a bidder's type increases, the relevance of the latter effect should increase. Theorem 5 below formalizes this intuition by showing that in general, there is no dominance relationship on the bidder payoffs from the interim perspective: there will always be some cutoff such that weaker types (below the cutoff) prefer to be the receiver, and higher types (above the cutoff) prefer to be the proposer.

Theorem 5. *There is no dominance relation on the interim payoffs for bidders 1 and 2; that is, for sufficiently low types, $U^1_{Int}(s) < U^2_{Int}(s)$, while for sufficiently high types, $U^1_{Int}(s) > U^2_{Int}(s)$.*

[24] Note that $\bar{s}_2(\bar{t}(\cdot))$ is strictly increasing for $s_1 < s_1^*$, and so $\hat{s}_1(s_2)$ is unique on this domain, while $\bar{s}_2(\bar{t}(s_1^*)) = 1$.

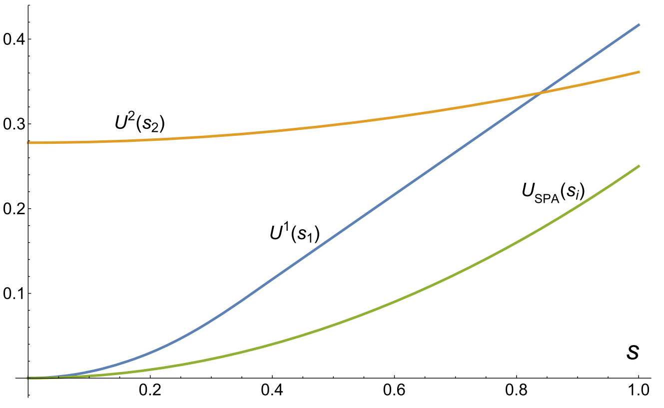

Fig. 3. Expected profits for bidders 1 and 2 ($U^1_{Int}(s_1)$ and $U^2_{Int}(s_2)$) compared with expected profits in a competitive second price auction with no collusion for the case of pure common values.

4.2. *Common value example*

To illustrate the consequences with regard to surplus distribution explored above more concretely, we return to the common-value setting introduced in Section 3. This will also allow us to get a sense of the relative magnitudes of the surplus distribution in three cases: no collusion, collusion with an uninformed third party, and collusion with signaling. Recall that the value to both bidders is just the average of their private signals, $(S_i + S_j)/2$, and the signals are distributed iid uniformly. The equilibrium strategies are $\bar{t}(s_1) = \min\{s_1, 1/3\}$ and $\bar{s}_2(t) = \min\{3t, 1\}$.

4.2.1. *Interim payoffs*

The expected payoffs to each bidder, conditional on her signal (equations (6) and (7)), become

$$U^1_{Int}(s_1) = \begin{cases} \frac{3s_1^2}{4}, & s_1 \le \frac{1}{3} \\ \frac{s_1}{2} - \frac{1}{12}, & s_1 > 1/3 \end{cases} \tag{8}$$

$$U^2_{Int}(s_2) = \int_0^{\frac{s_2}{3}} \left(\frac{s_2 - s_1}{2}\right) ds_1 + \int_{\frac{s_2}{3}}^{1} \min\left\{s_1, \frac{1}{3}\right\} ds_1 = \frac{10 + 3s_2^2}{36} \tag{9}$$

In a standard second price auction with no collusion stage, the bidders are symmetric, and the expected payoffs are simply $U^i_{SPA}(s_i) = s_i^2/4$. The plots of these functions are shown in Fig. 3. From the figure, it is clear that both bidders are always better off relative to the no collusion benchmark for every possible type. In addition, as seen in the figure, low types of bidder 2 are much better off than low types of bidder 1, but there is a point at which this relationship reverses, and for sufficiently high types, $U^1_{Int}(s_1)$ dominates $U^2_{Int}(s_2)$. It is also of interest to note that the critical point $\hat{s}$ where $U^1_{Int}(\hat{s}) = U^2_{Int}(\hat{s})$ is much greater than the type $s_1^* = 1/3$, where bidder 1 is sufficiently strong that all higher types begin to pool at t^*. That is, it is only a very small mass of types of bidder 1 who receive a higher expected payoff than the equivalent type of bidder 2.

Table 1
Distribution of ex-ante equilibrium payoffs under various scenarios.

	Bidder 1	Bidder 2	Seller
No collusion	0.085	0.085	0.33
Collusion with uninformed third party	0.25	0.25	0
Collusion with signaling	0.175	0.305	0.02

4.2.2. *Ex-ante payoffs*

In our model, we endow bidder 1 with all of the bargaining power to approach bidder 2. Both bidders do better than without collusion, but, from the interim perspective, it is not immediately clear whether it is advantageous to be the proposer or the receiver: high types prefer to propose, while low types prefer to receive. We can also analyze this question from the ex-ante perspective, to see how the total expected surplus is distributed between the bidders.

In a competitive auction with no collusion, both bidders are symmetric. The ex-ante expected payoff to bidder i is

$$\int_0^1 \int_0^{s_i} \frac{s_i - s_j}{2} ds_j ds_i = \frac{1}{12} \approx 0.08333.$$

The total ex-ante surplus is $\int_0^1 \int_0^1 \frac{s_1+s_2}{2} ds_1 ds_2 = 1/2$. Since the outcome is efficient, this implies that the ex-ante expected revenue to the seller is $1/2 - 2 \times 1/12 = 1/3$.

As discussed previously, in the UTP approach to modeling collusion in which the goal is maximization of the joint surplus of the bidding ring, the optimal collusive mechanism is trivial with common values: the cartel allocates the object according to some exogenous rule. Because reports do not affect the resulting allocation, there is no incentive for bidders to misreport. With common values, this allocation rule is clearly efficient. Assuming the ring opts for a symmetric scheme (e.g., flipping a coin), each bidder's ex-ante surplus is 1/4, and the seller receives 0 revenue.

A major assumption inherent in this result is that bidders commit to the cartel ex-ante. If bidders have private information when they decide upon collusion, this may not work. We model collusion taking place at the interim stage, but can still evaluate payoffs from the ex-ante stage. While the UTP approach finds that all agents receive the same ex-ante surplus, our bidders are not symmetric, and so we find a non-symmetric distribution. Integrating equations (8) and (9), we find that bidder 1's ex-ante payoff is 0.175 and bidder 2's expected payoff is 0.305. The distribution of ex-ante payoffs under various models is summarized in Table 1. Regarding the seller, the UTP approach to collusion would predict that collusion always occurs and that the colluding bidders are able to extract the entire surplus; in our model, as in Eső and Schummer (2004), collusion may not occur in equilibrium and the seller is able to retain some positive profits.

We know from Theorem 5 that being the proposer is better for high types, while being the receiver is better for low types (though both are better off than without collusion). From the ex-ante perspective, Table 1 shows more of the gains accrue to the receiver: she is on average more better off than the proposer. Looking at Fig. 3, there is a cutoff type above which it is better to be the proposer (as we know must be true from Theorem 5); however, this cutoff value is relatively high, which means that there is a much larger mass of bidders who prefer to be the receiver, and so being the receiver is better from the ex-ante perspective. A natural question is whether this ex-ante dominance is a general feature, or whether it is possible for the proposer to gain more than the receiver from the ex-ante perspective.[25] It turns out that it is indeed possible for the proposer to be better off, depending on parameters. The relevant parameter is the degree of correlation between signals, with more highly correlated signals tending to lead to higher ex-ante payoffs for the proposer. The intuition is that, when the signals are highly correlated, the proposer's own signal s_1 gives her a good deal of information about s_2, and so she can more precisely target her offer, and thus, keep a greater proportion of the surplus for herself.[26]

5. Outside competition

5.1. Interdependent values

In the previous discussion, the actual content of bidder 2's signal is irrelevant to bidder 1; all that matters is that bidder 2 stays out of the auction. However, if there are bidders outside of the cartel, the information of bidder 2 is now relevant to bidder 1 in updating her beliefs about the value of the object and refining her bid against the outsiders. In this section, we introduce a third bidder to study how competition from outside the cartel affects the distribution of surplus between the bidders and the seller.

We now have 3 bidders, named 1, 2 and 3, and a seller; bidder 1 is again the informed principal who can approach bidder 2 with a collusion proposal. For simplicity, we focus on the common value case from Section 4.[27] The signals are all distributed iid uniformly over [0, 1], and the value of the object is now[28]:

$$V(S_1, S_2, S_3) = \frac{S_1 + S_2 + S_3}{3}.$$

We assume that bidder 3 is not a member of the cartel, but is aware of the possibility of collusion between bidders 1 and 2, and observes who participates in the auction.[29] The next proposition identifies equilibrium bidding behavior. Note that if collusion is successful, bidder 1 knows both s_1 and s_2, while if collusion is unsuccessful, bidder 2 knows both s_1 (which she infers from the transfer offered by bidder 1) and s_2.

[25] I thank an anonymous referee for raising this question.

[26] One formal way to model this is to assume that the signals (s_1, s_2) are jointly distributed according to a copula, which allows for a simple parameterization of correlation, and also allows the equilibrium to be easily (numerically) solved. For example, if (s_1, s_2) are distributed according to a Gumbel copula with parameter 2, one can calculate that proposer will have a higher ex-ante surplus than the receiver (0.28 vs. 0.20).

[27] As we will see, equilibria will be qualitatively similar to what we solved for above. This qualitative structure will also hold allowing for a more general model of arbitrary interdependent values and affiliated signals, similar to Section 3. The algebra becomes much more complex and provides little additional insight, and so, for clarity, we focus on this simpler case.

[28] We normalize by 3, rather than 2, so that the total ex-ante surplus is the same as with two bidders, which will allow us to compare the ex-ante expected payoffs here with those before.

[29] An earlier draft of the paper considered a model in which bidder 3 is unaware of the potential collusion between bidders 1 and 2, and follows the standard second price auction equilibrium strategy. The equilibrium is qualitatively similar, though (as expected) bidder 3's equilibrium payoff is lower. I thank an anonymous referee for suggesting I look at the model presented here instead.

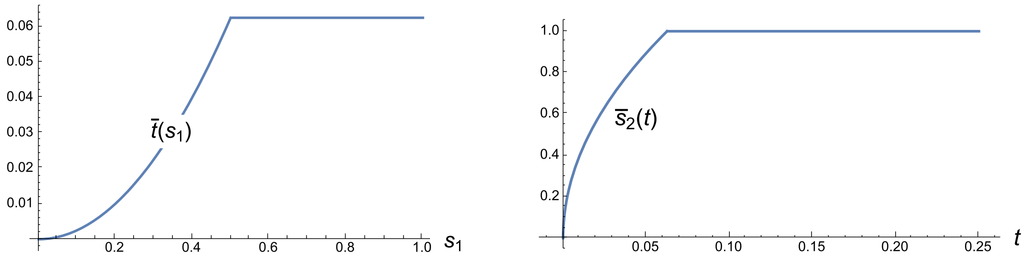

Fig. 4. Equilibrium strategies for bidders 1 (left) and 2 (right).

Proposition 6. *The following strategies are an equilibrium of the auction stage: in the continuation game after successful collusion, $\beta_1(s_1, s_2) = (s_1 + s_2)/2$ and $\beta_3(s_3) = s_3$; in the continuation game after unsuccessful collusion, $\beta_1(s_1) = s_1$, $\beta_2(s_1, s_2) = (s_1 + s_2)/2$, and $\beta_3(s_3) = s_3$.*

Moving back to the collusion stage, we continue to identify bidder 2's strategy with a cutoff function $\bar{s}_2(\cdot)$, and look for an equilibrium $(\bar{t}, \bar{s}_2)$ that is partially separating. In the separating region, we can write an equation analogous to equation (2) for bidder 1's utility if he is of true type s_1 and he mimics type $\hat{s}_1$ as:

$$\tilde{U}^1(\hat{s}_1, s_1) = \int_0^{\bar{s}_2(\bar{t}(\hat{s}_1))} [\Omega(s_1, s_2) - \bar{t}(\hat{s}_1)] ds_2.$$

This takes the same form as equation (2), only we replace $\omega(s_1, s_2)$ with $\Omega(s_1, s_2)$, which is bidder 1's expected payoff when collusion is accepted and he faces bidder 3 tomorrow using updated signals (s_1, s_2), defined as $\Omega(s_1, s_2) = \int_0^{(s_1+s_2)/2} \left[V(s_1, s_2, s_3) - \beta(s_3)\right] ds_3$. We can again use local IC to characterize $\bar{t}$ in terms of a differential equation $\bar{t}'(s_1) = g(s_1, \bar{t}(s_1))$ in the separating region, which must be solved numerically. We plot the equilibrium functions for bidder 1 and bidder 2 in Fig. 4.

The equilibrium is qualitatively very similar to that found before. Low types of bidder 1 separate, up to the point where offering more guarantees all types of bidder 2 will accept, beyond which all types pool. The more interesting question is how outside competition affects the distribution of surplus between the bidders and the seller, and thus, the profitability of collusion. Fig. 5 plots the interim expected profits for each of the three bidders in this equilibrium. Low types of bidder 2 continue to be relatively more better off than low types of bidder 1, though this again reverses for high types. Bidder 3 is also made better off when the other bidders collude. Comparing Fig. 5 with Fig. 3 (which is the analogous graph for the case of an all-inclusive cartel), we see that even though the graphs are the same qualitatively, the magnitudes are markedly smaller (note the scale of the y-axes on the two graphs are different). Thus, the presence of an additional bidder outside of the cartel makes the seller significantly better off. We can also calculate the ex-ante profits in the three bidder case, and compare to the two bidder model. The results are shown in Table 2. With 2 bidders, collusion greatly harms the profits of the seller. With three bidders, the surplus retained by the seller becomes much closer to what she would receive in a standard competitive auction with no collusion.

Some intuition for this can be found by comparing Fig. 4 with the equilibrium strategies in the 2 bidder case. We see that with a third bidder, the transfers offered by bidder 1 in the first stage

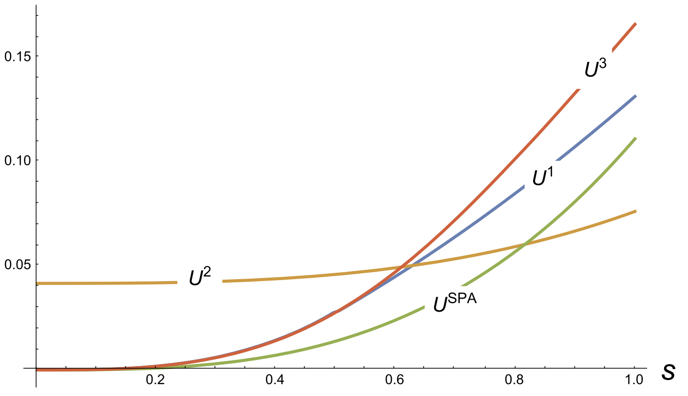

Fig. 5. Interim expected profits for bidders 1, 2, and 3 (U^1, U^2, and U^3, respectively) as a function of their observed signal and the (symmetric) interim expected profits in a competitive second price auction with no collusion (U^{SPA}). Compare with Fig. 3.

Table 2
Comparison of ex-ante equilibrium payoffs with 2 versus 3 bidders with and without collusion (rounded to three decimal places). The left-hand side is taken from Table 1.

	Two bidder model		Three bidder model	
	No collusion	Collusion	No collusion	Collusion
Bidder 1	0.085	0.175	0.028	0.042
Bidder 2	0.085	0.305	0.028	0.050
Bidder 3	–	–	0.028	0.048
Seller	0.33	0.02	0.417	0.355

decrease dramatically: the maximum transfer offered with outside competition is about 0.07, while in the two bidder case, it was 0.33. The introduction of a third bidder has two effects: first, colluding becomes more valuable to bidder 1, because she gets more information, which she can use to refine her bid. At the same time, collusion also becomes less valuable to bidder 1, as it is possible that she will pay bidder 2 and still lose the auction to bidder 3, and thus her bribe is “wasted”. We can attribute this to a type of “winner’s curse” (perhaps more accurately labeled a “briber’s curse”): when bidder 2 accepts the offer from bidder 1, this signals to bidder 1 that bidder 2 has a low signal, and the object is not very valuable. Thus, she must pay bidder 2 upfront, and still face bidder 3 in the auction. In equilibrium, this causes bidder 1 to decrease her initial transfer significantly compared to the case when there was no competition outside of the cartel.[30] Fig. 4 shows that this negative effect dominates. This causes a feedback loop in which a lower transfer means that bidder 2 accepts far less often, which results in a significantly more competitive auction and higher revenues for the seller.

[30] A similar effect arises in the 2 bidder case as well, as in equilibrium, the fact that bidder 2 accepts will once again cause bidder 1 to update her expected value for the object downward. However, it becomes more serious with a third bidder outside of the cartel, because not only must bidder 1 pay bidder 2 to stay out of the auction, but she in addition still must face bidder 3 in the auction itself.

Table 3
Comparison of ex-ante equilibrium payoffs with 2 versus 3 bidders with and without collusion under independent private values. To make accurate comparisons, results are reported as the proportion of the total possible ex-ante surplus accruing to each agent. (For two bidders, total possible ex-ante surplus is 2/3, while for three bidders, it is 3/4; the values in the "Collusion" columns sum to less than 1 because with private values, there is social inefficiency and a loss of surplus.)

	Two bidder model		Three bidder model	
	No collusion	Collusion	No collusion	Collusion
Bidder 1	0.250	0.360	0.111	0.128
Bidder 2	0.250	0.495	0.111	0.167
Bidder 3	–	–	0.111	0.112
Seller	0.500	0.104	0.67	0.591

5.2. *Private values*

For completeness, we also analyze the effects of introducing a third bidder under independent private values (IPV). A bidder's valuation is equal to her signal, and the signals are distributed iid uniformly on [0, 1]. We analyze the partially separating equilibrium of Theorem 1 (which reduces to the equilibrium of Eső and Schummer, 2004). The results are shown in Table 3.

A few insights can be gathered by comparing Tables 2 and 3. First, as expected, with independent private values, we continue to find that both bidders are better off under collusion, bidder 2 gains more than bidder 1, and these gains are reduced in the presence of outside competition. However, the overall gains to collusion are much less under private values; this holds both when the cartel is all inclusive, and when there is outside competition in the form of a third bidder, because under the IPV assumption, bidder 2's signal has no value to bidder 1, and thus there is no information-sharing motivation for collusion. Third, note that outside competition has a much greater impact on seller profits under common values relative to private values. Fourth, note that bidder 3 is once again made slightly better off when the other two bidders are colluding. The effect is small, but is in contrast to the UTP approach (e.g., Marshall and Marx, 2007). Last, in contrast to the common value case, bidder 3, while better off relative to no collusion, no longer gains more than bidder 1. These latter effects can again be attributed to the so-called "briber's curse" discussed above, which only arises under interdependent values. When values are interdependent, the briber's curse causes bidder 1 to lower her offer to bidder 2, which leads bidder 2 to reject more often. This means that it is more likely that there will be three bidders in the final auction, which increases the profits of the seller.

6. Conclusion

In this paper, we model collusion as a game in which one informed bidder must propose to another. This highlights important signaling aspects inherent in the mere act of offering to collude which are not present in standard models of collusion where the collusive mechanism is designed and executed by an uninformed third party. We characterize the unique continuous and partially separating equilibrium of the game when values are interdependent and signals affiliated. Analyzing the equilibrium outcomes, collusion is inefficient if there is a private component to valuations. At the same time, both bidders are better off than without it, and so the loss in efficiency is mostly borne by the seller. We study the distribution of surplus to the cartel bidders, and exhibit both costs and benefits to being the proposer of a collusive agreement: low types lose (relative to being the receiver) because of information leakage, while high types gain because of the opportunity to signal their strength. Outside competition significantly decreases the gains to colluding. Interestingly, the outside bidder can actually gain more than the proposing bidder in the cartel, which can be attributed to the fact that the outside bidder can free ride on the proposer's bribe. This is exacerbated by a newly-identified "briber's curse" whereby the fact that the receiver accepts the proposal means her signal is relatively low, which causes the proposer to downgrade her expectations before the ultimate auction.

While the signaling aspect of proposing does suggest some difficulties in initiating successful collusion agreements, this is not to suggest that collusion is something sellers (or governments) should be unconcerned about; indeed, our model predicts collusion will occur in equilibrium, as bidders are still better off colluding than not, which comes at the expense of seller revenue. However, it is likely in many cases that bidders do have some private information in hand before proposing or joining a cartel, and so it is important to understand the implications of modeling potential signaling more explicitly. This paper contributes to a small but growing literature that attempts to add more realistic features to collusion models while still keeping them analytically tractable, and we hope that richer models in this direction will continue to provide additional insights.

Appendix A. Proofs

Proof of Theorem 1. The proof consists of two claims.

Claim 7. *Equation 5 has a unique, continuous solution $\bar{t}(\cdot)$.*

Claim 8. *The strategies $(\bar{t}, \mathcal{A})$, together with appropriate beliefs, constitute an equilibrium.*

Proof of Claim 7. This part of the proof follows a similar argument to Eső and Schummer (2004). The differential equation (5) we are concerned with is

$$\bar{t}'(s_1) = \begin{cases} G(s_1, \bar{t}(s_1)), & h(s_1, \bar{t}(s_1)) < 1 \\ 0, & \text{else,} \end{cases}$$

where

$$G(s_1, \bar{t}(s_1)) = \frac{[\omega(s_1, h(s_1, \bar{t}(s_1))) - \bar{t}(s_1)][\omega_1(s_1, s_1) + \omega_2(s_1, s_1) - \omega_2(h(s_1, \bar{t}(s_1)), s_1)] f(h(s_1, \bar{t}(s_1))|s_1)}{\omega_1(h(s_1, \bar{t}(s_1)), s_1) F(h(s_1, \bar{t}(s_1))|s_1) - [\omega(s_1, h(s_1, \bar{t}(s_1))) - \bar{t}(s_1)] f(h(s_1, \bar{t}(s_1))|s_1)}$$

and the initial condition is $\bar{t}(0) = 0$.

We want to show that this equation has a solution, and that this solution is continuous. It is possible to show that $\bar{t}'(0) = \frac{\omega_1(0,0)[\omega_1(0,0)+\omega_2(0,0)]}{2\omega_1(0,0)-\omega_2(0,0)} > 0$.[31] Therefore, $0 < \bar{t}'(s_1) < \infty$ on some region $[0, \varepsilon)$. Now, the (local) inverse of $\bar{t}(s_1)$, denoted $S(\bar{t})$, is defined by

[31] This requires the use of l'Hopital's rule. The algebra is cumbersome, and was carried out with *Mathematica*.

$$S'(\bar{t}) = \frac{-[\omega(S(\bar{t}), h(S(\bar{t}),\bar{t})) - \bar{t}]f(h(S(\bar{t}),\bar{t})|S(\bar{t})) + F(h(S(\bar{t}),\bar{t})|S(\bar{t}))\omega_1(h(S(\bar{t}),\bar{t}), S(\bar{t}))}{[\omega(S(\bar{t}), h(S(\bar{t}),\bar{t})) - \bar{t}][\omega_1(S(\bar{t}), S(\bar{t})) + \omega_2(S(\bar{t}), S(\bar{t})) - \omega_2(h(S(\bar{t}),\bar{t}), S(\bar{t}))]f(h(S(\bar{t}),\bar{t})|S(\bar{t}))}$$
$$= \frac{F(h(S(\bar{t}),\bar{t})|S(\bar{t}))\omega_1(h(S(\bar{t}),\bar{t}), S(\bar{t}))}{[\omega(S(\bar{t}), h(S(\bar{t}),\bar{t})) - \bar{t}][\omega_1(S(\bar{t}), S(\bar{t})) + \omega_2(S(\bar{t}), S(\bar{t})) - \omega_2(h(S(\bar{t}),\bar{t}), S(\bar{t}))]f(h(S(\bar{t}),\bar{t})|S(\bar{t}))}$$
$$- \frac{1}{[\omega_1(S(\bar{t}), S(\bar{t})) + \omega_2(S(\bar{t}), S(\bar{t})) - \omega_2(h(S(\bar{t}),\bar{t}), S(\bar{t}))]}$$

with initial condition $S(0) = 0$. We claim that $S(\bar{t})$ is a well-defined, weakly increasing, and continuous function such that $S'(\bar{t}) > 0$ for almost all $\bar{t}$. First, note that $S'(0) = \frac{1}{\bar{t}'(0)} \in (0, \infty)$. Next, let $\bar{t}_0 > 0$ be such that $S'(\bar{t}_0) = 0$. We next show that if $S'(\bar{t}_0) = 0$, then $S''(\bar{t}_0) > 0$. At any such $\bar{t}_0$, we have the following, where, to shorten equations, we use the notation $J(x, y) = F(x|y)/f(x|y)$, and $\lambda(x, y) = \omega_1(y, y) + \omega_2(y, y) - \omega_2(x, y)$.[32]

$$S''(\bar{t}_0) = \frac{1 - \omega_2(S(\bar{t}_0), h(S(\bar{t}_0), \bar{t}_0))\big/\omega_1(h(S(\bar{t}_0), \bar{t}_0), S(\bar{t}_0)) + J_1(h(S(\bar{t}_0), \bar{t}_0), S(\bar{t}_0))}{J(h(S(\bar{t}_0), \bar{t}_0), S(\bar{t}_0))\omega_1(h(S(\bar{t}_0), \bar{t}_0), S(\bar{t}_0))\lambda(h(S(\bar{t}_0), \bar{t}_0), S(\bar{t}_0))} + \frac{\omega_{11}(h(S(\bar{t}_0), \bar{t}_0), S(\bar{t}_0))}{[\omega_1(h(S(\bar{t}_0), \bar{t}_0), S(\bar{t}_0)]^2\lambda(h(S(\bar{t}_0), \bar{t}_0), S(\bar{t}_0))}.$$

The regularity conditions give $J_1 \geq 0$ and $\omega_{11} \geq 0$. Further, we know $\omega_2(S(\bar{t}_0), h(S(\bar{t}_0), \bar{t}_0))\big/ \omega_1(h(S(\bar{t}_0), \bar{t}_0), S(\bar{t}_0)) < 1$ and $\lambda(h(S(\bar{t}_0), \bar{t}_0), S(\bar{t}_0)) > 0$.[33] Combining all of these facts, we conclude $S''(\bar{t}_0) > 0$ for any $\bar{t}_0$ where $S'(\bar{t}_0) = 0$. Thus, if $\bar{t}'(s_1)$ is ever infinite ($S'(\bar{t}_0) = 0$), then s_1 is only an inflection point of $\bar{t}(\cdot)$, and $\bar{t}(\cdot)$ continues with a positive and finite derivative to the right of s_1. This implies that equation (5) has a unique, continuous solution.

Proof of Claim 8. We first show that bidder 1's strategy is optimal, fixing bidder 2's strategy. We then show that bidder 2's strategy is optimal, given bidder 1's strategy and her updated beliefs about bidder 1's type. So, consider bidder 1 using function $\bar{t}(s_1)$. We start with the following lemma, whose proof can be found after the proof of the current theorem.

Lemma 9. *The function $\bar{t}(s_1)$ is strictly increasing on* $[0, t^*)$.

Now, our goal is to apply a comparative statics result of Athey (2002) (see also Athey, 1996). We first need the following definitions. A single-variable function $g : X \to \mathbb{R}$ satisfies the **single-crossing property (SCP)** if there exists $\inf X \leq x_0' \leq x_0'' \leq \sup X$ such that $g(x) < (\leq)0$ for all $x < (\leq)x_0'$, $g(x) = 0$ for all $x_0' < x < x_0''$, and $g(x) > (\geq)0$ for all $x > (\geq)x_0''$. A function of two variables $h : X \times Y \to \mathbb{R}$ satisfies the **two-dimensional single-crossing property (SCP-2)** if, for all $y_H > y_L$, $g(x) := h(x, y_H) - h(x, y_L)$ satisfies SCP. A single-variable function $g : X \to \mathbb{R}$ satisfies the **weak single-crossing property (WSCP)** if there exists an x_0 such that $g(x) \leq 0$ for all $x < x_0$ and $g(x) \geq 0$ for all $x > x_0$. A function of two variables $h : X \times Y \to \mathbb{R}$, where $X, Y \subseteq \mathbb{R}$, satisfies the **weak two-dimensional single crossing property (WSCP-2) in** (x, y) if $g(x) := h(x, y_H) - h(x, y_L)$ satisfies WSCP for all $y_H > y_L$.

Now, it is clearly dominated for bidder 1 to offer $t > t^*$ (see footnote 38). Thus, given $\bar{s}_2(t)$ for bidder 2, we only need to check

[32] Again, the algebra was carried out using *Mathematica*.

[33] For the last one, note that $g(x, y) = \omega(x, y) - \omega(y, y)$ is decreasing in y: $g_2(x, y) = \omega_2(x, y) - \omega_1(y, y) - \omega_2(y, y) < 0$. Then, $\lambda(x, y) = -g_2(x, y) > 0$.

$$\bar{t}(s_1) \in \arg\max_{t\in[0,t^*]} \tilde{U}(s_1,t)\ \forall s_1$$

where $\tilde{U}(s_1,t)$ is defined as[34]

$$\tilde{U}(s_1,t) = \int_{s_2} v(s_1,s_2,t)dF(s_2|s_1) \tag{10}$$

with

$$v(s_1,s_2,t) = [\omega(s_1,s_2)-t]\mathbb{I}(s_2<\bar{s}_2(t)) + [\omega(s_1,s_2)-\omega(s_2,s_2)]\mathbb{I}(\bar{s}_2(t)\le s_2<s_1). \tag{11}$$

We first show that, fixing s_1, bidder 1's payoff function v satisfies WSCP-2 in (s_2,t). To do so, we show that the returns to offering $t_H > t_L$, defined as $g(s_1,s_2) := v(s_1,s_2,t_H) - v(s_1,s_2,t_L)$, satisfy WSCP in s_2. By definition, $\bar{s}_2(t_L) < \bar{s}_2(t_H)$. Thus, the function g can be written piecewise as:

$$g(s_1,s_2) = \begin{cases} t_L - t_H, & s_2 < \bar{s}_2(t_L) \\ \omega(s_1,s_2) - t_H - [\omega(s_1,s_2)-\omega(s_2,s_2)]\mathbb{I}(s_2<s_1), & \bar{s}_2(t_L) \le s_2 < \bar{s}_2(t_H) \\ 0, & \bar{s}_2(t_H) \le s_2 \end{cases}.$$

It is simple algebra to check that $g(s_1,s_2)$ satisfies WSCP in s_2, and therefore v satisfies WSCP-2 in (s_2,t). Intuitively, g is negative (and constant) when s_2 is sufficiently low that both t_L and t_H would induce bidder 2 to accept the transfer; g is strictly increasing on the range over which offering t_H induces bidder 2 to accept, but t_L does not; and g is 0 when s_2 is sufficiently high that neither t_L nor t_H would induce bidder 2 to accept. Further, it can similarly be checked that $g(s_1,s_2)$ is piecewise continuous and nondecreasing in s_1. By affiliation, $f(s_2|s_1)$ is log-supermodular. We can thus apply Lemma 5, Extension (i) of Athey (2002) (see also Theorem 3.4 of Athey, 1996) to conclude $\tilde{U}(s_1,t)$ satisfies SCP-2.

We next show that $\tilde{U}(s_1,t)$ satisfies the following (which, when combined with SCP-2, Milgrom (2004) refers to as the **smooth single crossing differences condition**): for all $\delta \ge 0$, $\tilde{U}_2(s_1,t)=0 \implies \tilde{U}_2(s_1-\delta,t)\le 0$ and $\tilde{U}_2(s_1+\delta,t)\ge 0$.

First, calculate $\tilde{U}_2(s_1,t)$:

$$\tilde{U}_2(s_1,t) = f(\bar{s}_2(t)|s_1)\left[(\min\{\omega(s_1,\bar{s}_2(t)),\omega(\bar{s}_2(t),\bar{s}_2(t))\}-t)\,\bar{s}_2'(t) - \frac{F(\bar{s}_2(t)|s_1)}{f(\bar{s}_2(t)|s_1)}\right].$$

Note that the sign of $\tilde{U}_2$ is determined by the sign of the term in brackets, and further, the term in brackets is increasing in s_1 (because ω and $\bar{s}_2$ are both increasing, and $\partial(F(s_2|s_1)/f(s_2|s_1))/\partial s_1 \le 0$ by log supermodularity of the conditional distribution function $F(s_2|s_1)$, which is an implication of affiliation). This implies that $\tilde{U}(s_1,t)$ satisfies the smooth single crossing differences condition.

Finally, let $\pi(s_1) = \tilde{U}(s_1,\bar{t}(s_1))$ be the equilibrium payoff of type s_1, and note that by the envelope theorem, $\pi(s_1) = \int_0^{s_1} \tilde{U}_1(\xi,\bar{t}(\xi))d\xi$.[35] Theorem 4.2 of Milgrom (2004) then delivers that $\bar{t}(s_1) \in \arg\max_{t\in[0,t^*]} \tilde{U}(s_1,t)$; that is, $\bar{t}(s_1)$ is incentive compatible.

[34] This is simply another way to write bidder 1's payoff explicitly as a function of his type s_1 and action t, taking bidder 2's strategy $\bar{s}_2(t)$ as given.

[35] It can be checked directly that $\tilde{U}_2(s_1,\bar{t}(s_1))\bar{t}'(s_1)=0$. This implies that $\pi'(s_1)=\tilde{U}_1(s_1,\bar{t}(s_1))$, and so $\pi(s_1) = \int_0^{s_1}\tilde{U}_1(\xi,\bar{t}(\xi))d\xi$.

Thus, we have shown that, fixing $\bar{s}_2(t)$, the strategy $\bar{t}(s_1)$ is optimal for bidder 1. Now, we show that $\bar{s}_2(t)$ is indeed optimal for bidder 2.

First, we show that if bidder 2 knows the value of s_1 for sure, the optimal acceptance set takes the form of an interval $\mathcal{A}(t) = [0, \bar{s}_2(t)]$. When bidder 2 knows bidder 1's type is s_1, her payoff from the auction is $\mathbb{I}(s_1 < s_2)[\omega(s_2, s_1) - \omega(s_1, s_1)]$.[36] Thus, it is optimal to accept an offer of t if

$$t \geq \mathbb{I}(s_1 < s_2)[\omega(s_2, s_1) - \omega(s_1, s_1)].$$

The RHS is obviously increasing in s_2, and thus, if the inequality holds for any s_2, it also holds for all $s_2' \leq s_2$. Therefore, bidder 2 playing a strategy of $\mathcal{A}(t) = [0, \bar{s}_2(t)]$ is indeed optimal conditional on knowing s_1.[37]

Given Lemma 9, when bidder 2 observes some $t < t^*$, she believes bidder 1's type is $\bar{t}^{-1}(t)$ with probability 1. Therefore, her best response (by definition) is $\bar{s}_2(t) = h(\bar{t}^{-1}(t), t)$. If $t = t^*$, bidder 2 updates her beliefs and places all weight on $[s_1^*, 1]$, in which case it is simple to show that the best response is $\bar{s}_2(t^*) = 1$. If $t > t^*$, let bidder 2's beliefs be the same as if she had observed $t = t^*$. Then, $\bar{s}_2(t) = 1$ is a best response to any $t > t^*$ as well.[38]

Therefore, the proposed strategies do indeed constitute an equilibrium. □

Proof of Lemma 9. We show that $\bar{t}'(s_1) > 0$ whenever $h(s_1, \bar{t}(s_1)) < 1$, which implies that $\bar{t}(s_1)$ is strictly increasing, and hence, invertible, on this domain. The proof of part (i) of Theorem 1 shows that at $s_1 = 0$, we have $\bar{t}'(0) = \frac{\omega_1(0,0)[\omega_1(0,0)+\omega_2(0,0)]}{2\omega_1(0,0)-\omega_2(0,0)} > 0$. So, consider $0 < s_1 < h(s_1, \bar{t}(s_1))$. We must show that the numerator of (5) is strictly positive. The only potentially nontrivial part is to show $\omega(s_1, h(s_1, \bar{t}(s_1))) - \bar{t}(s_1) > 0$ for $s_1 > 0$. Define a function $g(s_1) = \omega(s_1, h(s_1, \bar{t}(s_1))) - \bar{t}(s_1)$, and consider its derivative:

$$g'(s_1) = \omega_1(s_1, h(s_1, \bar{t}(s_1))) + \omega_2(s_1, h(s_1, \bar{t}(s_1)))[h_1(s_1, \bar{t}(s_1)) + h_2(s_1, \bar{t}(s_1))\bar{t}'(s_1)] \\ - \bar{t}'(s_1).$$

Algebra shows (using the expression for $\bar{t}'(0)$ given above) that $g'(0) > 0$. So, $g(s_1) > 0$ over some small range $s_1 \in [0, \varepsilon)$. Further, note that for any $\hat{s}_1 > 0$, if $g(\hat{s}_1) = 0$, then $\bar{t}'(\hat{s}_1) = 0$, which further implies that $g'(\hat{s}_1) > 0$. Thus, we can conclude (see, e.g., the Ranking Lemma of Milgrom, 2004, page 124) that $g(s_1) > 0$ for all $s_1 > 0$, which is what we set out to show. □

Proof of Theorem 2. The proof is broken down into two parts.

Part (i): (Pooling at 0) If $\bar{t}(0) > 0$, then $\bar{t}(s_1) = \hat{t}$ for all $s_1 \in [0, \hat{s}_1]$ for some $\hat{s}_1 > 0$.

Let $\bar{t}(0) = t_0 > 0$. Assume instead that there is separation at 0. Upon observing an offer of t_0, bidder 1 perfectly reveals himself as type $s_1 = 0$. Thus, bidder 2 will accept his offer if and only if $t_0 \geq \omega(0, s_2)$. Thus, if type 0 of bidder 1 were to make such an offer, his expected payoff

[36] If $s_1 = s_2$, there is a 50/50 chance she will win the auction, but even if she does, her payoff is 0.

[37] As noted above, for the remainder of the paper, we will simply identify bidder 2's strategy with her cutoff function $\bar{s}_2(t)$.

[38] For an out-of-equilibrium transfer, bidder 2's beliefs can be anything that supports the equilibrium. However, note that no matter what bidder 2 believes and what her response to an action $t > t^*$ is, bidder 1 will never offer such a t, because she can always induce the same action from bidder 2 by offering a lower transfer (this follows because $\bar{t}([0, t^*]) = [0, 1]$). Thus, the equilibrium is essentially uniquely defined, in the sense that the outcome is independent of the beliefs chosen for bidder 2.

is $\int_0^{h(0,t_0)} [\omega(0, s_2) - t_0]\, dF(s_2|s_1 = 0)$. Since we just showed that $t_0 \geq \omega(0, s_2)$, this integral is clearly negative, and bidder 1 can profitably deviate by offering $t = 0$.

The first part shows that if bidder 1 of type 0 offers more than 0 in equilibrium, then the transfer function must begin with pooling at some fixed $\hat{t}$ on some domain $[0, \hat{s}_1]$ for some $\hat{s}_1 > 0$. The next part shows that if this is true, then all types of bidder 1 must pool at the same transfer $\hat{t}$. The intuition is that, if there were separation to the right of $\hat{s}_1$, then some small mass of types to the left of $\hat{s}_1$ could profitably deviate by mimicking some type $\hat{s}_1 + \epsilon$. To do so, they offer a transfer larger than $\hat{t}$, but this is more than offset by the fact that doing so separates them from the weaker types, thereby inducing a strictly positive mass of additional types of bidder 2 to accept.

Part (ii): (Complete pooling) If $\bar{t}(0) = \hat{t} > 0$, then $\bar{t}(s_1) = \hat{t}$ for all $s_1 \in [0, 1]$.

By part (i), we know that there must be pooling at $\hat{t}$ on some non-degenerate region $[0, \hat{s}_1]$. We show that, in fact, $\hat{s}_1 = 1$. Assume not. By continuity, there must be some $\hat{\hat{s}}_1 > \hat{s}_1$ such that $\bar{t}(s_1)$ is increasing on $(\hat{s}_1, \hat{\hat{s}}_1]$, with $\lim_{s_1 \to \hat{s}_1} \bar{t}(s_1) = \hat{t}$. Consider some small $\epsilon > 0$ and let $t_\epsilon = \bar{t}(\hat{s}_1 + \epsilon)$. Now, consider type $\hat{s}_1$. His equilibrium strategy is to offer $\hat{t}$. Upon seeing $\hat{t}$, bidder 2 updates his beliefs that bidder 1's type is distributed between $[0, \hat{s}_1]$. Define $\hat{s}_2$ to be the type of bidder 2 who is just indifferent between accepting and rejecting $\hat{t}$.[39] Recall that $h(s_1, t)$ is the type of bidder 2 who is indifferent between accepting and rejecting, conditional on perfectly observing bidder 1's type as s_1. If bidder 1 of type $\hat{s}_1$ deviates to $\hat{t}_\epsilon$, then the change in his expected payoff is $\int_{\hat{s}_2}^{h(s_1+\epsilon,t_\epsilon)} [\omega(s_1, s_2) - t_\epsilon] dF(s_2|s_2 \leq h(s_1+\epsilon, t_\epsilon)) + \int_0^{h(s_1+\epsilon,t_\epsilon)} [-t_\epsilon + \hat{t}] dF(s_2|s_2 \leq h(s_1+\epsilon, \epsilon))$. Now, note that on the relevant domain, $\omega(s_1, s_2) > t_\epsilon > \hat{t}$. By continuity, as $\epsilon \to 0$, $t_\epsilon \to \hat{t}$ and $h(s_1 + \epsilon, t_\epsilon) \to h(\hat{s}_1, \hat{t}) > \hat{s}_2$, and so, for $\epsilon > 0$ small enough, this is positive, and hence, a profitable deviation. □

Proof of Theorem 4. In a competitive auction (no ex-ante collusion), the interim expected payoff to bidder i is

$$U^i_{SPA}(s_i) = \int_0^{s_i} [\omega(s_i, s_{-i}) - \omega(s_{-i}, s_{-i})] dF(s_{-i}|s_i) \tag{12}$$

In the collusion game, if bidder 1 deviates to offering 0, bidder 2 always rejects and his payoff is $\int_0^{s_1} [\omega(s_1, s_2) - \omega(s_2, s_2)] dF(s_2|s_1)$, which is exactly $U^1_{SPA}(s_1)$. Therefore, his equilibrium payoff of the collusion game must be weakly higher than $U^1_{SPA}(s_1)$.

For bidder 2, write his interim payoff in the collusion game as:

$$U^2_{Int}(s_2) = \int_0^1 \left[\mathbb{I}(s_1 \leq \hat{s}_1(s_2))(\omega(s_2, s_1) - \omega(s_1, s_1)]) + \mathbb{I}(s_1 > \hat{s}_1(s_2))\bar{t}(s_1) \right] dF(s_1|s_2). \tag{13}$$

For $s_1 \leq \hat{s}_1(s_2)$, the integrands of (12) and (13) are the same. For $\hat{s}_1(s_2) < s_1 \leq s_2$,[40] the integrand of (12) is less than the integrand of (13). For $s_1 > s_2$, the integrand of (12) is 0, and the integrand of (13) is strictly positive. Thus, $U^2_{Int}(s_2) > U^2_{SPA}(s_2)$ for all s_2. □

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[39] That is, $\hat{s}_2$ solves is the value of s_2 that solves $\int_0^{\hat{s}_1} [\omega(s_2, s_1) - \omega(s_1, s_1)] dF(s_1|\hat{t}) = \hat{t}$.

[40] Recall that $\hat{s}_1(s_2)$ is the lowest type of bidder 1 who, in equilibrium, induces bidder 2 to accept his equilibrium transfer when of type s_2 (defined implicitly as $\hat{s}_1(s_2) = \min\{s_1 : \bar{s}_2(\bar{t}(\hat{s}_1)) = s_2\}$).By definition, $\hat{s}_1(s_2) \leq s_2$.

Proof of Theorem 5. To prove this theorem, we show the following: $0 = U^1_{Int}(0) < U^2_{Int}(0)$ and $U^1_{Int}(1) > U^2_{Int}(1)$. Continuity then implies that for all s sufficiently low, we have $U^1_{Int}(s) < U^2_{Int}(s)$, while for all s sufficiently high, we have $U^1_{Int}(s) > U^2_{Int}(s)$.

For the first inequality, if $s_1 = 0$, then $\bar{t}(s_1) = 0$, and bidder 1 is sure to lose the auction, receiving a payoff of 0; on the other hand, if $s_2 = 0$, bidder 2 still sometimes achieves a strictly positive payoff (when bidder 1 offers her a strictly positive transfer and she accepts), and so $U^1_{Int}(0) < U^2_{Int}(0)$.

For the second inequality, first, note that type s_1^* strictly prefers to offer t^* to offering 0, which gives

$$U^1_{Int}(s_1^*) = \int_0^1 [\omega(s_1^*, s_2) - t^*] dF(s_2|s_1^*) > \int_0^{s_1^*} [\omega(s_1^*, s_2) - \omega(s_2, s_2)] dF(s_2|s_1^*).$$

This can be rewritten as

$$E_{S_2}[h(s_1^*, s_2)|S_1 = s_1^*] > t^* \tag{14}$$

where $h(s_1^*, s_2) = \min\{\omega(s_2, s_2), \omega(s_1^*, s_2)\}$. Then,

$$\begin{aligned}
&U^1_{Int}(1) - U^2_{Int}(1)\\
&= \int_0^1 [\omega(1, s_{-i}) - t^*] dF(s_{-i}|s_i = 1)\\
&\quad - \left[\int_0^{s_1^*} [\omega(1, s_{-i}) - \omega(s_{-i}, s_{-i})] dF(s_{-i}|s_i = 1) + \int_{s_1^*}^1 t^* dF(s_{-i}|s_i = 1)\right]\\
&= \int_{s_1^*}^1 [\omega(1, s_{-i}) - \omega(s_1^*, s_{-i}) - t^*] dF(s_{-i}|s_i = 1) + \int_0^{s_1^*} \omega(s_{-i}, s_{-i}) dF(s_{-i}|s_i = 1)\\
&\quad + \int_{s_1^*}^1 \omega(s_1^*, s_{-i}) dF(s_{-i}|s_i = 1) - \int_0^1 t^* dF(s_{-i}|s_i = 1)\\
&> 0
\end{aligned}$$

where the inequality follows because the first integral is weakly positive because $\omega(1, s_1^*) - \omega(s_1^*, s_1^*) = t^*$ and ω is supermodular, while the sum of the last three is (strictly) positive by equation (14) and affiliation. □

Proof of Proposition 6. First consider the continuation game following successful collusion, where the only auction participants are bidders 1 and 3. Define $\tilde{s} = (s_1 + s_2)/2$ to be the average of the two signals that bidder 2 has observed. Given bidder 3's strategy, bidder 1's payoff from submitting a bid of b is $\mathbb{I}\{b \geq s_3\} \times [(2\tilde{s} + s_3)/3 - s_3] = \mathbb{I}\{b \geq s_3\} \times [2/3 \times (\tilde{s} - s_3)]$. Her bid does not affect the price she pays, only whether she wins or not. Since her payoff upon winning is positive if and only if $\tilde{s} \geq s_3$, submitting a bid of $\tilde{s}$ is a best response. As for bidder 3, her payoff from submitting a bid b is $\mathbb{I}\{b \geq \tilde{s}\} \times [(2\tilde{s} + s_3)/3 - \tilde{s}] = \mathbb{I}\{b \geq \tilde{s}\} \times [(s_3 - \tilde{s})/3]$. Similarly, her bid does not affect the price she pays, only whether she wins or loses. Her payoff is positive if and only if $s_3 \geq \tilde{s}$, and so, given the strategy of bidder 1, submitting a bid of s_3 is a best response. Next, consider the continuation game in which collusion does not occur, and so there are 3 bidders at the auction. Note that if this stage is reached, all bidders know that bidder 2 has rejected, and so $s_2 > s_1$, which further implies that $\tilde{s} > s_1$. First, consider bidder 1. Her payoff is $\mathbb{I}\{b \geq \max\{\tilde{s}, s_3\}\} \times [(2\tilde{s} + s_3)/3 - \max\{\tilde{s}, s_3\}]$. Thus, she wins the auction only if her bid is $b \geq \max\{\tilde{s}, s_3\}$. It is easy to check that submitting such a bid gives her a (weakly) negative for any realization (s_1, s_2, s_3), and so submitting a bid equal to her signal s_1 is a best response. Similarly, consider bidder 2. Note that at any s_1 such that bidder 2 rejects $\bar{t}(s_1)$ in the collusion stage, $\bar{t}(s_1)$ is strictly increasing. Thus, bidder 2 can infer s_1 with certainty, i.e., she observes $\tilde{s} = (s_1 + s_2)/2$. Given the strategies of the other bidders, bidder 2's optimization problem is to choose a bid b that solves $\max_b \int_0^{s_1} [V(s_1, s_2, s_3) - s_1] ds_3 + \int_{s_1}^{b} [V(s_1, s_2, s_3) - s_3] ds_3$. Solving this maximization problem shows that submitting a bid of $\tilde{s} = (s_1 + s_2)/2$ is optimal. For bidder 3, when she observes bidder 2 participating in the auction, she knows that $s_2 > s_1$, and thus the highest outside bid she will face is $\tilde{s} = (s_1 + s_2)/2$. Bidder 3's payoff from submitting a bid b is once again $\mathbb{I}\{b \geq \tilde{s}\} \times [(2\tilde{s} + s_3)/3 - \tilde{s}] = \mathbb{I}\{b \geq \tilde{s}\} \times [(s_3 - \tilde{s})/3]$. Since she does not affect her price, only whether she wins or loses, submitting a bid of s_3 is a best response. □

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