Desirable Rankings: A New Method for Ranking Outcomes of a Competitive Process

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Abstract

We consider the problem of aggregating individual preferences over alternatives into a social ranking. A key feature of the problems that we consider—and the one that allows us to obtain positive results, in contrast to negative results such as Arrow's Impossibililty Theorem—is that the alternatives to be ranked are outcomes of a competitive process. Examples include rankings of colleges or academic journals. The foundation of our ranking method is that alternatives that an agent *desires*—those that they have been rejected by—should be ranked higher than the one they receive. We provide a mechanism to produce a social ranking given any preference profile and outcome assignment, and characterize this ranking as the unique one that satisfies certain desirable axioms.

1 Introduction

What college should a student choose? Should an assistant professor be granted tenure? For a wide range of important decisions, we rely on external rankings of quality to help us make a good decision.

One standard approach to constructing a ranking is to use a formula based on indicators of quality. An important example is the *US News and World Report* rankings of colleges, which receive a great deal of media attention and have been shown to influence the decisions of college applicants (Bowman and Bastedo, 2009; Griffith and Rask, 2007). While the precise formula behind this ranking is opaque, inputs include quality indicators such as the acceptance rate (percentage of applicants who are accepted) and the yield rate (percentage of those admitted who enroll).

Similarly, deciding whether an assistant professor's body of research is sufficient to justify granting tenure or determining what share of state funding a university should receive are strongly influenced by rankings of journals.[1] In turn, journal rankings are typically based on measures such as acceptance rates or citation counts.

Both of the examples above are susceptible to Goodhart's Law: when a measure becomes a target, it ceases to be a good measure. Indeed, there is evidence that universities purposefully solicit applications from students they know will be rejected in order to lower their admissions rates, as well as reject highly qualified applicants that they fear may choose another school so as not to harm their matriculation rates (Toor, 2000; Golden, 2001; Belkin, 2019). Similarly, a journal interested in improving its journal impact factor (JIF) has a strong incentive to manipulate its acceptance rate and citation count, and there are well-known strategies for doing so.[2]

[1] For example, research funding is allocated to British universities in part using the "Research Excellence Framework" which aims to take a metrics based approach to measuring research impact.

[2] See, for instance, "A user's guide to inflated an manipulated impact factors" (Ioannidis and Thombs, 2019); "Authorship and citation manipulation in academic research" (Fong and Wilhite, 2017); "Editors? JIF-boosting stratagems–Which are appropriate and which not?" (Martin, 2016); and "Games academics play and their consequences: how authorship, h-index and journal impact factors are shaping the future of academia" (Chapman et al., 2019).

Rather than relying on seemingly "objective" measures such as acceptance rates or citation counts, which can be gamed, an alternative approach is to aggregate individual rankings to produce a social ranking. For instance, which colleges students have chosen in previous years can be informative for the decisions of the current cohort. Information about the best academic journals can be obtained from the order in which researchers submit to (and are rejected from) them. Of course, not all agents will agree on the answers to these questions, and so the question remains of how to aggregate diverse individual preferences into an aggregate ranking. Many results in social choice theory (e.g., Arrow's Impossibility Theorem (Arrow, 1950), the Gibbard-Satterthwaite Theorem (Gibbard, 1973; Satterthwaite, 1975)), speak to the difficulty of such preference aggregation.

In this paper, we propose a new method of ranking alternatives. For concreteness, and with the above application in mind, we refer throughout to the alternatives to be ranked as *colleges*. However, our method is general, and can be applied to many other settings, such as ranking public elementary schools, medical residency programs, or academic journals. The key feature that will allow us to obtain positive results in contrast with the impossibility results discussed above is that the alternatives we are ranking are outcomes of a competitive process: a student matriculates at a college after being accepted by it (and rejected by others); a research article appears in a journal after it has been rejected from other journals that the authors prefer.

Consider a student i who attends a college B but prefers and was rejected from college A (alternatively, a researcher who submits an article to journal B after having been rejected at A, or a doctor who is matched to residency program B but prefers another program A). The fact that college admissions are the outcome of a process in which students make application decisions and colleges make acceptance/rejection decisions allows us to infer two pieces of information. First, student i *desires* A (over B); but second, it also reveals that college A prefers its student, say j, to student i, because A rejected i in favor of j. In a sense, A agrees with i's assessment that it is the better college: A would not trade, even if offered. This is the basis of our aggregate rankings: an object that an agent desires should be ranked higher.

We begin by introducing a series of axioms on rankings that formalize the idea that any college a student prefers should be ranked higher. We call rankings that satisfy these axioms *desirable rankings*. Our main contribution is an algorithm to produce desirable rankings. We call the algorithm the *delayed trading cycles (DTC)* algorithm, and the ranking produced the *DTC ranking* of colleges. To construct a DTC ranking, given some outcome of the college admissions market, we start by identifying the largest subset of colleges C' such that no student assigned to a school outside of C' desires any school in C'. Since no student at a college outside of C' desires any college in C', the colleges in C' are ranked lower than any college outside of C'.[3] After removing the colleges in C', we then identify the largest subset of remaining colleges, C'', such that no student assigned to a college outside of C'' desires any college in C''. Notice that since any $c'' \in C''$ was not removed in the first step, there must be some student assigned to a college in C' that desires c''; thus, we rank the colleges in C'' ahead of those in C', but below all other colleges. Continuing in this manner, we obtain a ranking of all colleges. We characterize the DTC ranking as the unique ranking that satisfies the desirability axioms.

There are other methods that can be used to produce rankings of alternatives. One typical approach used by economists is to make inferences using revealed preference: if a student is admitted to colleges A and B and chooses A, then we infer that she prefers A to B. This is the basis of the rankings introduced by Avery et al. (2013). Each student i is viewed as a tournament among the colleges they were admitted to, and the college i selects among these is the "winner" of the tournament for student i. Colleges accumulate points for each student tournament that they "win", and rankings are determined in a manner analogous to how chess or tennis players are ranked based.

There is an important conceptual distinction between revealed preference rankings and desirable rankings. Revealed preference makes inferences by looking "down" an

[3] All of the schools in C' are ranked equally. The final output of the DTC algorithm is an ordered partition of the colleges, and we refer to any two colleges that are ranked the same as belonging to the same *tier*.

agent's preference list: it uses information about colleges that are *worse* than the one an agent is assigned to form an aggregate ranking. Desirability on the other hand, looks "up" an agent's preference list: it uses information about the objects an agent *desires* to construct an aggregate ranking.

We think there are several advantages to desirability. First, desirability is less likely to make incorrect inferences in the presence of idiosyncratic individual preferences. Consider a professor who leaves university A to accept a position at university B. Revealed preference infers that B is better than A. However, it is common for a professor to leave a relatively high-ranked university for a lower-ranked one. For instance, if the professor is returning to her home country or she has a two-body problem, she might make this choice, and yet, we would not want to conclude that this choice reveals the relative quality of the institutions; rather, it is likely the result one person's idiosyncratic preferences. Desirability internalizes this; in particular, desirability only ranks college B over A when a professor at A wants a position at B, and B is unwilling to hire her.

A second advantage of desirability over revealed preference is that it is more general. For revealed preference to be applicable, (at least some) students must receive multiple offers. In settings where assignments are done through a centralized process, such as in medical residency matching, public school choice, or university admissions in many countries (e.g., China, India, Turkey), the students/doctors do not receive more than one offer. Even for some decentralized markets, revealed preference is not always applicable. For example, to use revealed preference to rank academic journals, a researcher would need to submit an article to multiple journals simultaneously, which is considered unethical.[4] On the other hand, it is easy to discern which journals an author desires: it is those that she submitted the article to (and was rejected from) prior to the one at which it was accepted. Therefore, even though the assignment of articles to journals is decentralized, it is not possible to use revealed preference to rank journals. Desirability may be used to rank journals, as well

[4] An exception is law review journals, which often permit simultaneous submissions.

as any of the other above applications.

2 Model

There is a set of students $I = \{i_1, \ldots, i_N\}$ and a set of colleges $C = \{c_1, \ldots, c_N\}$. Each college c has a capacity of $q_c = 1$, and a strict ranking of the students $\succ_c$. We use $\succsim_c$ for the corresponding weak ranking, i.e., $i \succsim_c j$ if either $i \succ_c j$ or $i = j$. Each student has a strict preference relation P_i over the colleges. We use R_i for the corresponding weak ranking, i.e., $c \; R_i \; c'$ if either cP_ic' or $c = c'$. An *outcome* μ is an assignment of students to colleges; formally, $\mu : I \cup C \to I \cup C$ is a function such that $\mu_i \in C$ for all $i \in I$, $\mu_c \in I$ for all $c \in C$, and $\mu_i = c$ if and only if $\mu_c = i$.

Throughout, we fix an outcome μ. We will provide a method to determine a ranking taking any μ as an input, and so we do not make any assumptions on how μ is determined. For instance, μ may arise from a fully decentralized process as in the U.S. college admissions market, a more structured process such as in academic journal submissions, in which an article can be sent to only one journal at a time, or fully a centralized process similar to college admissions in China, National Resident Matching Program, or public school choice.

A **ranking** of the colleges is a weak ordering on the set C, denoted $\trianglerighteq$. If $a \trianglerighteq b$ but $b \not\trianglerighteq a$, then we write $a \triangleright b$ and say that a is ranked **higher** than b. When $a \trianglerighteq b$ and $b \trianglerighteq a$, we write $a \simeq b$. Any ranking $\trianglerighteq$ induces an ordered partition of the colleges $\Pi^{\trianglerighteq} = \{\Pi_1^{\trianglerighteq}, \Pi_2^{\trianglerighteq}, \ldots, \Pi_K^{\trianglerighteq}\}$ that can be defined recursively as:

$$\Pi_1^{\trianglerighteq} = \{c \in C : c \trianglerighteq c' \text{ for all } c' \in C\}$$
$$\Pi_k^{\trianglerighteq} = \left\{c \in C \setminus \cup_{k'=1}^{k-1} \Pi_{k'}^{\trianglerighteq} : c \trianglerighteq c' \text{ for all } c' \in C \setminus \cup_{k'=1}^{k-1} \Pi_{k'}^{\trianglerighteq}\right\}$$

In words, $\Pi_1^{\trianglerighteq}$ is the set of colleges that are ranked the highest, $\Pi_2^{\trianglerighteq}$ is the set of colleges that are ranked higher than all others except those in $\Pi_1^{\trianglerighteq}$, etc. We refer to the colleges in $\Pi_k^{\trianglerighteq}$ as the **tier-k colleges**. Similarly, we refer to the students assigned to these colleges, $\mu(\Pi_k^{\trianglerighteq}) = \{i : \mu_i \in \Pi_k^{\trianglerighteq}\}$, as the **tier-$k$ students**. Define the function $\tau : I \cup C \to \mathbb{N}$ such that $\tau(x) = k$, where k is the tier to which agent x (which may be a student or a college) belongs. If $k < \ell$, then any college c such that $\tau(c) = k$ is ranked higher than any c' such that $\tau(c') = \ell$; a similar remark applies to students.

When no confusion arises, we will also use the term "ranking" to refer to the partition/tier structure $\Pi^{\trianglerighteq} = \{\Pi_1^{\trianglerighteq}, \Pi_2^{\trianglerighteq}, \ldots, \Pi_K^{\trianglerighteq}\}$ induced by $\trianglerighteq$. Phrases such as "college c is ranked k^{th}" or "c is a k^{th} ranked college" are interpreted as $\tau(c) = k$.

3 Axioms

If a student receives her favorite school, it may be because she is a very good student attending a very good school; alternatively, it may be that she prefers a less good school simply because she has idiosyncratic preferences (e.g., the school is close to home), and no other students are interested in it. In the latter case, she is likely to be admitted to this school as her first choice, yet we would not want the aggregate ranking to rank this school highly. Therefore, instead of basing our ranking on which school a student attends, we base our ranking on which schools a student has been rejected by but prefers to her assignment, i.e., the schools that the student desires.

Definition 1. *Student i* ***desires*** *college a if $aP_i\mu_i$.*

Using this definition, we present our first axiom.

Axiom 1 (Strong Axiom of Desire)**.** A ranking $\trianglerighteq$ satisfies the **strong axiom of desire (SAD)** if for all students i and all colleges a that i desires, we have $a \triangleright \mu_i$.

In words, the strong axiom of desire says that if a student desires a school, then it must be ranked higher than her assignment. While this is a very natural (and desirable) condition, it is possible that no ranking will satisfy it. This is shown in the next example.

Example 1. Suppose there are two states, A and B, and each state has two state universities, $\{A^{good}, A^{bad}, B^{good}, B^{bad}\}$. There are two students, i and j. Student i lives in state A, and student j lives in state B. All students agree that within a state, the good school is preferred to the bad school. Both students wish to get away from home and prefer any college out of state to any college in state. Specifically, for student i, $B^{good}\ P_i\ B^{bad}\ P_j\ A^{good}\ P_i\ A^{bad}$ while for student j, $A^{good}\ P_j\ A^{bad}\ P_j\ B^{good}\ P_j\ B^{bad}$. However, it is harder to get into a college out of state than it is to get into a college in state, and both students are good enough to get into any college in state but no college out of state. Therefore, the stable assignment μ in this instance is for i to attend A^{good} and j to attend B^{good}.[5]

There is no ranking that satisfies strong axiom of desire: since i desires B^{good}, SAD requires $B^{good} \rhd A^{good}$, but since j desires A^{good}, SAD also requires $A^{good} \rhd B^{good}$, a contradiction.

In the example above, the outcome μ is Pareto inefficient from the student's perspective; indeed, any Pareto inefficient assignment will not admit a ranking that satisfies SAD. Note that a Pareto inefficient assignment (from the perspective of the students) is due to the admissions criteria of the colleges.

At the same time, Example 1 suggests the following intuitive ranking as the "correct" one:

$$A^{good} \simeq B^{good} \rhd A^{bad} \simeq B^{bad}$$

This corresponds to the following tiers:

School	$\tau(c)$
A^{good}	1
B^{good}	1
A^{bad}	2
B^{bad}	2

[5]An assignment μ is *stable* if there is no student i and college c such that $c\ P_i\ \mu_i$ and $i \succ_c \mu_c$.

Notice that student i desires B^{bad}, and B^{bad} is ranked higher than her assignment, A^{good}; however, there is another school she desires even more, B^{good}, that is ranked the same as her assignment. This motivates our next axiom, which is a weakening of SAD. To state it, we first need the following definition.

Definition 2. *Student i **strongly desires** college a if aP_ib for all $b \in C$ such that $\tau(b) = \tau(\mu_i)$.*

In words, a tier-k student strongly desires a college c if she prefers c to any of the tier-k colleges.

Axiom 2 (Weak Axiom of Desire)**.** Ranking $\trianglerighteq$ satisfies the **weak axiom of desire (WAD)** if for every student i and every college a that i strongly desires, we have $a \triangleright \mu_i$.

Strong desire is clearly a strengthening of desire: if i strongly desires a, then she prefers a not only to her assignment (i.e., i desires a), but she also prefers a to any college that is ranked the same as her assignment. Correspondingly, the strong axiom of desire implies the weak axiom of desire, but the converse does not hold. The reason we focus on WAD is two-fold. First, as shown above, a ranking satisfying SAD may not exist. Second, in the next section, we will present an algorithm to calculate a ranking and characterize it, and from the perspective of characterization, using a weaker axiom is preferable.

In our analysis, it will be convenient to use alternative formulations of WAD. While trivial to prove, we state them as a lemma, for ease of reference.

Definition 3. *Given a student i and a subset of colleges $C' \subseteq C$, we define*

$$fav_i(C') = \{c \in C' : cP_ic' \text{for all } c' \in C'\}$$

*to be i's **favorite** school in C'.*

Lemma 1. *The following are equivalent to the Weak Axiom of Desire:*

1. For all k, all tier-k students i, and all colleges c in tiers lower than k ($\tau(c) > k$):

$$\text{fav}_i(\Pi_k^{\trianglerighteq}) \; P_i \; c.$$

2. For all k and all colleges $c \in \Pi_k^{\trianglerighteq}$:

$$\text{fav}_{\mu_c}(\Pi_k^{\trianglerighteq}) = \text{fav}_{\mu_c}(\cup_{\ell \geq k} \Pi_\ell^{\trianglerighteq}).$$

In words, this says that any tier-k student must prefer at least one of the tier-k schools to any of the lower tier schools. We provide some examples to illustrate WAD.

Example 2. Let i be a student, and suppose there are three schools: a, b, c. In the tables below, the left column is i's preferences, while the right gives the tier of the corresponding school. We use a box to indicate i's assigned school. For example, in the first scenario below, $a \; P_i \; b \; P_i \; c$; i is assigned to b; and the ranking we are considering, Π^1, ranks a, b, and c first, second, and third, respectively.

P_i	τ^1
a	1
$\boxed{\text{b}}$	2
c	3

Not surprisingly, τ^1 satisfies WAD as it "agrees" with i's own preferences. Next consider:

i	τ^2
$\boxed{\text{a}}$	3
b	1
c	2

Ranking τ^2 satisfies WAD vacuously: student i ranks a first and τ^2 ranks a last, but i does not desire (or strongly desire) any college. Next consider τ^3:

i	τ^3
a	1
b	2
$\boxed{\text{c}}$	1

Ranking τ^3 also satisfies WAD. Student i desires b and yet b is ranked lower than her assignment, c. Yet there is an object she desires even more than b, a, that is ranked the same as her assignment.

i	τ^4
a	1
b	3
$\boxed{\text{c}}$	2

Ranking τ^4 violates WAD. Student i desires b, yet b is ranked lower than her assignment or any college ranked the same as her assignment (in this case there are no other such colleges).

Many rankings satisfy WAD. For instance, the trivial ranking in which all colleges are ranked the same always satisfies WAD. However, this is not a useful ranking in general, and intuitively is "correct" only if each student is getting their first-choice. When this is true, then no college is desired by any student, and so no college should be ranked higher than any other. When this is not the case—the more common scenario—there is some student who is not getting their first choice. This implies that there is some college A that is the first choice of more than one student, and another college B that is not the first choice of any student. Thus, we posit that in this case, college A should not be ranked equivalent to B. Our next axiom generalizes this idea to any tier.

Definition 4. *Given a ranking $\trianglerighteq$, consider the tier-k colleges $\Pi_k^{\trianglerighteq}$ and tier-k students $\mu(\Pi_k^{\trianglerighteq})$. For any $a \in \Pi_k^{\trianglerighteq}$, let $D_k(a) = |\{i \in \mu(\Pi_k^{\trianglerighteq}) : \mathrm{fav}_i(\Pi_k^{\trianglerighteq}) = a\}|$. If $D_k(a) > q_a$, then we say college a is* ***overdemanded****, and if $D_k(a) < q_a$, we say say that college*

*a is **underdemanded**.*[6] *If no college $a \in \Pi_k^{\trianglerighteq}$ is overdemanded or underdemanded, then we say that tier k is **balanced**.*

Axiom 3 (Balancedness)**.** Ranking $\trianglerighteq$ is **balanced** if every tier $\Pi_k^{\trianglerighteq}$ is balanced.

Note that if no two schools are ranked the same (each tier is a singleton), then balancedness holds trivially.

Our final axiom is based on the following observation. Consider a large market and a student i who is unusual in the following sense. Every other student considers i's college c to be the worst college, but i thinks that c is the best college. Since i does not desire any college, WAD does not require that c be ranked below any college. Moreover, as no other student even weakly desires c, WAD does not require us to rank c above any college. In particular, it is consistent with WAD to have c ranked first, last, or anywhere in between. Yet, it seems intuitively clear that i's preferences are idiosyncratic, and the "right" ranking is to have c be ranked last. This would be the converse of WAD: to be ranked higher, an object should be desired. Our final axiom is a weaker version of this condition that requires justification for a college to be in tier k rather than a lower tier $k+1$. Justification is trivial for all any college in the lowest tier. Beyond this, justification should be transitive in the following sense. Suppose we have justified college c's tier-k ranking, and let i be the student attending c. Consider another college $\tilde{c}$. If i weakly prefers $\tilde{c}$ to any of the tier-k colleges, then it is justified to rank $\tilde{c}$ at least k^{th}. If i strictly prefers $\tilde{c}$ to any of the tier-k colleges, then it is justified to rank $\tilde{c}$ at least $(k-1)^{th}$. This suggests an inductive definition of which rankings have been justified.

Definition 5. *Consider a ranking $\trianglerighteq$ with K tiers, $\Pi^{\trianglerighteq} = \{\Pi_1^{\trianglerighteq}, \ldots, \Pi_K^{\trianglerighteq}\}$ and a tier-k college $c \in \Pi_k^{\trianglerighteq}$. College c's ranking is **justified** if any of the following conditions are satisfied:*

1. *k is the lowest ranking: $k = K$;*

[6] Note that in our setting, if there exists an overdemanded college, then by the pigeonhole principle, there must also exist an underdemanded college.

2. *a tier-$(k+1)$ college $\tilde{c}$'s ranking has been justified, and $c \ P_{\mu_{\tilde{c}}} \ c'$ for every tier-$(k+1)$ college c';*

3. *a tier-k college $\tilde{c}$'s ranking has been justified, and $c \ R_{\mu_{\tilde{c}}} \ c'$ for every tier-k college c'.–*

Axiom 4 (Justified)**.** Ranking $\trianglerighteq$ is **justified** if every college's ranking is justified.

We will refer to any ranking $\trianglerighteq$ that is Balanced, Justified, and satisfies the Weak Axiom of Desire as a **desirable ranking**. Our main result is to show that there is a unique desirable ranking of colleges.

Theorem 1. *There is a unique desirable ranking of colleges.*

We prove Theorem 1 in the next section.

4 Characterizations of Ranking Algorithms

4.1 Top Trading Cycles Rankings

In this section, we introduce a class of algorithms to determine a ranking $\trianglerighteq$ for any outcome μ and preference profile P. We then provide a characterization result of this class using the weak axiom of desire and balancedness. Our ranking algorithm is based on the top trading cycles (TTC) algorithm of Shapley and Scarf (1974).

Definition 6. *Consider an outcome μ and a set of colleges C'. A **top trading cycle of** C' is a list of distinct colleges $\chi = (c_1, c_2, \ldots, c_n)$ such that for all k, $c_k \in C'$ and*

$$c_{k+1} = \mathrm{fav}_{\mu_{c_k}}(C'),$$

*where the subscripts k are taken modulo n, i.e., $c_{n+1} = c_1$. Given a set of colleges C' and a subset $C'' \subseteq C'$, we call C'' a **set of cycles of** C' if C'' can be decomposed* *into top trading cycles of C'.*[7]

Decomposing colleges into top trading cycles is a way of ranking the colleges.

Definition 7. *A ranking $\trianglerighteq$ is a **TTC-ranking** if for every tier $\Pi_k^{\trianglerighteq}$ and every school $c \in \Pi_k^{\trianglerighteq}$, c belongs to a top trading cycle of $C \setminus \cup_{j<k}\Pi_j^{\trianglerighteq}$.*

We emphasize that a tier $\Pi_k^{\trianglerighteq}$ is not required to contain all of the top trading cycles. A TTC-ranking corresponds to a sequence of selecting cycles when implementing TTC. As there is not a unique cycle selection rule for TTC, there is not a unique TTC-ranking. Rather, TTC-rankings are a class of rankings.

Theorem 2 (Broad Characterization)**.** *A ranking of colleges satisfies WAD and balancedness if and only if it is a TTC-ranking.*

Proof. First we prove that a TTC ranking satisfies the the weak axiom of desire and balancedness. Suppose $\Pi = [\Pi_k]$ is a TTC ranking. Then, each Π_k is the union of top trading cycles of $C \setminus \cup_{j<k}\Pi_j$. Consider any tier Π_k and any $c \in \Pi_k$. College c is part of a top trading cycle; let c' denote the college student μ_c is pointing to in this cycle. By definition, c' is μ_c's favorite remaining college. Specifically, $\text{fav}_{\mu_c}(C \setminus \cup_{j<k}\Pi_j) = c'$. In particular, for any $c'' \in \cup_{j\geq k}\Pi_k = C \setminus \cup_{j<k}\Pi_j$, $c' \; R_{\mu_c} \; c''$. Indeed, if $c'' \neq c'$, then $c' \; P_{\mu_c} \; c''$. Therefore, the weak axiom of desire is satisfied. There are no two colleges $c, c' \in \Pi_k$ where both μ_c and $\mu_{c'}$ have the same favorite college as Π_k consists of distinct top trading cycles. Therefore, assigning each student i such that $\mu_i \in \Pi_k$ to i's favorite school in Π_k is a perfect assignment. Therefore, balancedness is satisfied as well.

We prove uniqueness by induction. Suppose $\Pi = [\Pi_k]$ is a ranking that satisfies the weak axiom of desire and balancedness. For the base step, we consider Π_1. Consider college $c \in \Pi_1$ and let $i = \mu_c$. By the weak axiom of desire, $\text{fav}_i(\Pi_1) = \text{fav}_i(C)$. In words, if we ask i to point to her favorite college, she points to a college in Π_1.

[7] Note that these cycles would necessarily be distinct.

However, if we ask each student assigned to a college in Π_1 to point to her favorite college in Π_1, no two students point to the same college, by balancedness. Therefore, we are able to decompose Π_1 into top trading cycles. For the inductive step, suppose each $\Pi_1, \dots, \Pi_{k-1}$ is a sequence of top trading cycles, and consider Π_k and let C' denote $C \setminus \cup_{j<k} \Pi_j$. Suppose i is a student where $\mu_i \in \Pi_k$. Similar to the base step, by the weak axiom of desire, $\text{fav}_i(\Pi_k) = \text{fav}_i(C')$. Therefore, when we ask each remaining student to point to her favorite remaining school, each student i where $\mu_i \in \Pi_k$ points to a college in Π_k. Indeed, if we ask this same student to point to her favorite school in Π_k, she must point to the same school. By balacedness, no two students whose assignments are in Π_k point to the same college, or else there would not exist a perfect assignment. Therefore, Π_k must be the disjoint union of top trading cycles. $\square$

4.2 Delayed Trading Cycles

TTC rankings are an entire class of rankings that depend on the cycle selection rule. As shown in the previous section, the class of TTC rankings are characterized by WAD and balancedness. This still leaves many possible TTC rankings. Adding our third axiom that, in addition to WAD and balancedness, requires each college's ranking to be justified will pin down the cycle selection rule and give a unique desirable ranking.

To construct the cycle selection rule that will give a justified ranking, we work from the bottom up, by first identyfing the colleges that should be ranked last. A college should be ranked last if no student strongly desires it. This motivates the following definition. We say a matching is **perfect** if every student receives her favorite school.

Definition 8. *Given an outcome μ, a set of colleges $C' \subseteq C$ is **isolated perfectly** if*

1. *(isolated) for every student i such that $\mu_i \notin C'$ and every college $c \in C'$, $\mu_i \; P_i \; c$.*

2. (perfectly) there exists a perfect matching $\nu : \mu(C') \to C'$.

When convenient, we will refer to an isolated-perfectly set of colleges as an **I.P. set**. I.P. sets are a generalization of a related concept of (Kesten and Kurino, 2016), who call a college underdemanded if $\mu_i R_i c$ for all students i.[8] If a college c is underdemanded in the sense of (Kesten and Kurino, 2016), then $\{c\}$ is an I.P. set. In general, an underdemanded college may not exist (even for a stable assignment). However, the next example shows that an I.P. set may exist even if there are no underdemanded schools.

Example 3. Suppose there are two students, i and j, and two colleges, c and c', each with unit capacity. The student preferences and the schools ranking of students are below.

i	j		c	c'
c	c'		j	i
c'	c		i	j

The assignment

$$\mu = \begin{pmatrix} i & j \\ c' & c \end{pmatrix}$$

has no underdemanded school, yet the set $\{c, c'\}$ is I.P. Condition (1) is satisfied trivially, and assigning i and j to c and c', respectively, is a perfect assignment.

We will show that there is a unique maximum I.P. set.

Definition 9. *Given an outcome μ, a set of colleges $C' \subseteq C$ is the* ***maximum, isolated-perfectly*** *set of schools if C' is I.P. and for any set of colleges C'' that is I.P. under μ, $C'' \subseteq C'$.*

[8] Note that this is a different usage of the term underdemanded than we use in the balancedness axiom.

The next lemma shows that the union of two I.P. sets is also I.P.. This immediately implies that there is a well-defined maximum I.P. set, as the union of all I.P. sets is an I.P. set.

Lemma 2. *Consider an outcome μ, and suppose $A, B \subseteq C$ are two I.P. sets of colleges. Then $A \cup B$ is also I.P.*

Proof. Suppose A and B are I.P.. Condition (1) is straightforward. Consider any i such that $\mu_i \notin A \cup B$. Since $\mu_i \notin A$ and since A is I.P., $\mu_i \ P_i \ c'$ for any $c' \in A$. Symmetrically, $\mu_i \ P_i \ c'$ for any $c' \in B$. Therefore, $\mu_i \ P_i \ c'$ for any $c' \in A \cup B$.

We now show condition (2). Define $\nu^A : \mu(A) \to A$ as follows: for each i such that $\mu_i \in A$, set $\nu_i^A = \text{fav}_i(A)$. Define $\nu^B : \mu(B) \to B$ analogously. Since A and B are I.P., both ν^A and ν^B are proper matches.

Consider a student $i \in \mu(A \setminus B)$. Since $i \notin B$ and B is I.P., $\mu_i \ P_i \ c$ for any $c \in B$. In particular, $\text{fav}_i(A) \in A \setminus B$ as $\mu_i \in A \setminus B$. Since ν^A is defined as $\text{fav}_i(A)$, $\nu_i^A \in A \setminus B$.

Next, consider a student $i \in A \cap B$. We will show $\nu_i^A \in A \cap B$ (note that if $A \cap B = \emptyset$, then this holds trivially). Suppose for contradiction that $\nu_i^A \in A \setminus B$. Since $\nu^A(A \setminus B) \subset A \setminus B$, by the pigeonhole principle, there must be a school $c \in A \setminus B$ where $|\nu_c^A| > |\mu_c^A|$. However, this violates the stability of the match μ. Each school is acceptant, and if $|\nu_c^A| > |\mu_c|$, then c is willing to accept more students than under μ (ν^A is a proper match). As there is a student $i \in \nu_c^A$ that prefers c to μ_i, i and c block μ, a contradiction. Therefore, for each $i \in \mu(A \cap B)$, $\nu_i^A \in A \cap B$. We summarize (where points ii. and iii. follow from a symmetric argument for the set B):

i if $i \in \mu(A \setminus B)$, then $\nu_i^A \in A \setminus B$.

ii if $i \in \mu(A \cap B)$, $\nu_i^A \in A \cap B$.

iii if $i \in \mu(B \setminus A)$, then $\nu_i^B \in B \setminus A$.

iv if $i \in \mu(A \cap B)$, $\nu_i^B \in A \cap B$.

Consider a student $i \in \mu(A \cap B)$. Since $\mathrm{fav}_i(A) \in A \cap B$, $\mathrm{fav}_i(A) = \mathrm{fav}_i(A \cap B)$. Similarly, $\mathrm{fav}_i(B) = \mathrm{fav}_i(A \cap B)$. Therefore, $\mathrm{fav}_i(A) = \mathrm{fav}_i(A \cap B) = \mathrm{fav}_i(B)$. In particular, if $i \in \mu(A \cap B)$, then $\nu_i^A = \nu_i^B$. Therefore, the following match is well-defined.

$$\nu_i^{A \cup B} = \begin{cases} \nu_i^A & \text{if } \mu_i \in A \\ \nu_i^B & \text{if } \mu_i \in B \end{cases}$$

By construction, $\nu^{A \cup B}$ is a perfect match from $\mu(A \cup B)$ to $A \cup B$, and condition (2) is satisfied. $\square$

Corollary 1. *Any outcome μ contains a unique, maximum I.P. set.*

For convenience, given any initial outcome μ, we let μ^{TTC} denote the outcome obtained after running TTC on μ. Although there always exists a unique, maximum I.P. set, this set may be empty; however, we will show μ^{TTC} always has a nontrivial, maximum I.P. set.

Lemma 3. *For any outcome μ, the TTC outcome μ^{TTC} contains a nontrivial I.P. set.*

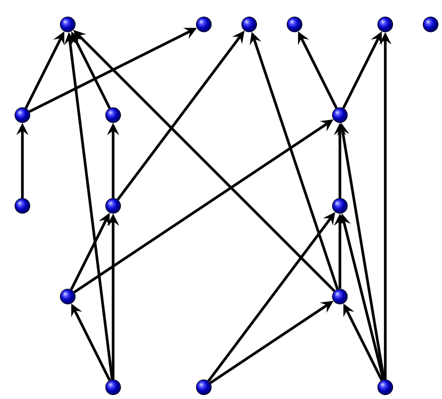
Proof. Starting with assignment μ, let $\chi_1, \dots, \chi_n$ be any sequence of sets of Top Trading Cycles. Specifically, each χ_k is the union of top trading cycles on the set $C \setminus \cup_{j<k} \chi_j$. We claim that χ_n is I.P.. Consider student i such that $\mu_i \notin \chi_n$. Student i is part of an earlier trading cycle than any cycle in χ_n. As i could have pointed to one of the schools in χ_n but did not, by WAD, $\mu_i^{TTC} \; P_i \; c$ for any $c \in \chi_n$. $\square$

We can now introduce our main algorithm for calculating rankings. Given an outcome μ, the algorithm first calculates μ^{TTC} and then iteratively removes the maximum I.P. set. The colleges removed in any given step are ranked immediately above all of the colleges removed in the previous step. We call this algorithm **Delayed Trading Cycles (DTC)**, and the ranking of colleges it induces the **Delayed Trading Cycles ranking**.

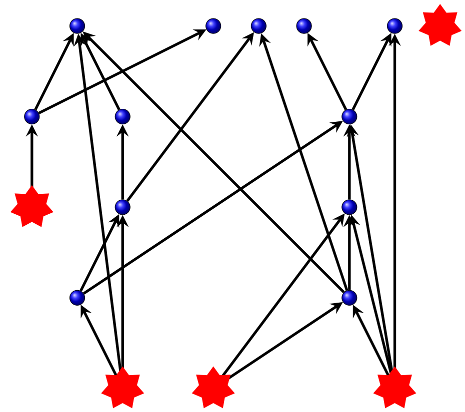
Definition 10. *Given an outcome μ, let μ^{TTC} be the outcome obtained by running TTC on μ. Define M_1 to be the maximum, I.P. set of μ^{TTC}. In general, set $C_k = C \setminus \cup_{1 \leq l \leq k} M_l$ and if $C_k \neq \emptyset$, then define M_{k+1} to be the maximum I.P. set of $C \setminus C_k$. If $C_k = \emptyset$, then stop. Let $M_1, M_2, \ldots, M_K$ be the resulting partition.*

*The **Delayed Trading Cycles (DTC) ranking** of the colleges, $\trianglerighteq^{DTC}$ is given as follows: for any two colleges $a, b \in C$, where $a \in M_\ell$ and $b \in M_k$, $a \trianglerighteq b$ if and only if $\ell \geq k$. The tier-k colleges are $\Pi_k^{DTC} = M_{K-k+1}$, and the DTC partition is $\Pi^{DTC} = \{\Pi_1^{DTC}, \ldots, \Pi_K^{DTC}\}$.*

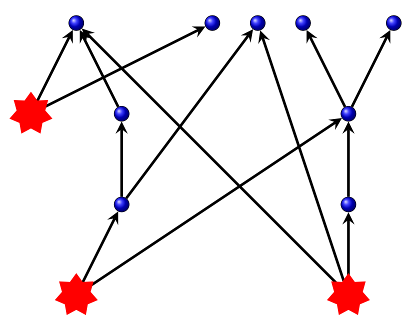
Delayed Trading Cycles is easiest to understand pictorially. In the graph below, each node represents a trading cycle, and we will refer to nodes and cycles interchangeably. There is a directed arrow from node C_1 to node C_2 if there is a student in cycle C_1 that desires one or more of the schools in C_2. Note that by construction, the cycles from TTC produce a directed tree. No student in a top trading cycle desires any schools outside the cycle, so there cannot be an edge from an earlier cycle to a later cycle. Suppose TTC produces the following graph:



The maximum I.P. set is simply the nodes at the "bottom" of the tree. Namely, cycles that do not have any cycle pointing to them. We indicate the maximum I.P. set with large red stars in the graph below.



Delayed Trading Cycles ranks these schools last and then removes them. This produces the following graph (with the maximum I.P. set again indicated by large red stars).



Under DTC, these "last" cycles are ranked second-to-last, removed, and the process is repeated. We now prove Theorem 1 by characterizing the DTC ranking as the unique desirable ranking.

Theorem 3 (Main Characterization)**.** *The DTC ranking is the unique ranking that is Balanced, Justified, and satisfies WAD.*

Proof. As DTC is a TTC ranking, by Theorem 2 it satisfies WAD and Balancedness. We show by induction that DTC is Justified. The lowest-tier schools are justified by definition. Now, consider any tier-k which is not the lowest tier, and suppose that the ranking of every college ranked $k+1$ or lower has been justified. The tier-k colleges can be decomposed into cycles of μ^{TTC}. By construction, every one of these cycles is "pointed" to by a $k+1$ cycle. More precisely, for every cycle $\tilde{C} \subset \Pi_k$, there exists a college $c \in \tilde{C}$ and tier-$(k+1)$ student i such that i strongly desires c. By the inductive hypothesis, all tier-$(k+1)$ colleges have justified rankings. Therefore, c's ranking is justified. Without loss of generality, label the cycle $\tilde{C} = (c, c_2, \ldots, c_n)$. We have determined that c's ranking is justified. By the definition of TTC, $\mu(c) = i$'s favorite school in tier-k is c_2. Therefore, c_2's ranking is justified. Student $\mu(c_2)$ justifies c_3's ranking, and so on. As $\tilde{C}$ was an arbitrary cycle in Π_k, we conclude all of the k rankings are justified.

Now suppose a ranking $\Pi = \{\Pi_1, \ldots, \Pi_n\}$ is balanced, justified, and satisfies WAD. From Theorem 2, Π is a TTC ranking, and in particular, partition Π_n can be decomposed into top trading cycles. As a reminder, let μ^{TTC} denote the assignment resulting from applying TTC to μ. We will call a cycle $\tilde{C}$ a "last" cycle if for every student i such that $\mu(i) \notin \tilde{C}$, $\mu^{TTC}(i)\ P_i\ c$ for every $c \in \tilde{C}$ (in our construction of DTC, these are the cycles that are not pointed to by any other cycle). Π is a TTC ranking, so the partition Π_n consists of trading cycles that were removed at the same time. As this is the last partition, these are all "last" cycles. Let X denote the last partition under DTC (the lowest ranked colleges). By the definition of DTC, X is the maximum I.P. set of μ^{TTC}, and X is the union of "last" cycles. Therefore, $\Pi_n \subseteq X$. Next, consider an arbitrary "last" cycle $tildeC$. By definition, for every student i such that $\mu(i) \notin \tilde{C}$, $\mu^{TTC}(i)\ P_i\ c$ for every $c \in \tilde{C}$. In particular, such a student does not justify the ranking of any college in $\tilde{C}$. By assumption, the ranking of every college in C is justified, but as none of these rankings are justified by a student not in C, this is only possible if the colleges in $\tilde{C}$ are ranked last. Therefore, for every "last" cycle $\tilde{C}$, $\tilde{C} \subseteq \Pi_n$. This implies that $X \subseteq \Pi_n$, and therefore, $X = \Pi_n$.

The inductive argument is identical. Suppose for k is a non-negative integer and that the partitions $\Pi_{n-k}, \ldots, \Pi_n$ each correspond to the DTC partitions. Now consider partition Π_{n-k-1}. We will call cycle $\tilde{C}$ a "last" cycle if for every student i such that $\mu(i) \notin \tilde{C} \cup_{0 \leq l \leq k} \Pi_{n-l}$, $\mu^{TTC}(i)\ P_i\ c$ for every $c \in \tilde{C}$. These are the last possible cycles of TTC if we removed the colleges $\Pi_{n-k}, \ldots, \Pi_n$. As a reminder, the union of these cycles constitutes the $n-k-1$ partition of DTC. We label the union of "last" cycles by the set X. As Π is a TTC ranking, each cycle in Π_{n-k-1} must be a last cycle. Therefore, $\Pi_{n-k-1} \subseteq X$. Moreover, Π_n must contain every "last" cycle $\tilde{C}$. The rankings of the schools in $\tilde{C}$ are not justified by any student not in $\tilde{C} \cup \Pi_{n-k}, \ldots, \Pi_n$. Therefore, the colleges in $\tilde{C}$ must be ranked $n-k-1$ or lower, or else their rankings would not be justified. This implies that $\tilde{C} \subseteq \Pi_{n-k-1}$, and we conclude that $X \subseteq \Pi_{n-k-1}$. So indeed, $X = \Pi_{n-k-1}$. $\square$

5 Conclusion

We consider the problem of how to rank alternatives that are the outcome of a competitive process, such as rankings of colleges or academic journals. We introduce a new paradigm for constructing rankings that is based on the notion of desirability: alternatives that an agent desires (relative to what she receives) should be ranked higher. We introduce several axioms that formalize the notion of desirability, and build an algorithm, the Delayed Trading Cycles algorithm, that can be used to calculate desirable rankings. Further, we characterize the DTC ranking as the unique ranking desirable ranking.

In future work, we plan to explore other properties of desirable rankings. For instance, we have discussed how desriable rankings are less prone to errors in inference due to idiosyncratic agent preferences. Formally, one could build a model in which student preferences over colleges are a combination of a common component (the college's overall quality) and an private component (a student's idiosyncratic preferences), and ask whether desirable rankings uncover the true quality ranking of colleges. This could also be compared with the more standard revealed preference approach of Avery et al. (2013). Additionally, standard methods of calculating rankings (acceptance rates, citation counts) are problematic because they are susceptible to gaming. Notice that under a desirable ranking, for a college to be ranked higher, it must be desired by a student at a lower-ranked school. Thus, if a college wants to raise its rank, it must increase its quality to become a more attractive college. Thus, desirable rankings should provide better incentives for colleges to invest in quality improvement as opposed to strategies that artificially infltate their rankings. Formalizing this and showing that desirable rankings are less susceptible to gaming on the part of the colleges is an interesting question for future work.

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