Prices and Prestige: The Connection Between Desirable Rankings and Competitive Equilibria*

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Abstract

The need for rankings of objects is important in many economic contexts. We consider a house allocation model where each agent owns a unique house, and the goal is to create a ranking of the houses. One natural approach is to use an analogue of competitive equilibrium, where the ranking plays the role of prices, as in Richter and Rubinstein (2015). A drawback of this approach is that such competitive equilibrium rankings are not unique. We show that the axiomatic approach of desirable rankings introduced by Aryal et al. (2024) solves this ambiguity and produces a unique refinement of competitive equilibrium rankings we call the *true prestige*. Along the way, we establish a fundamental connection between desirable rankings and competitive equilibrium rankings by giving market-based interpretations of Aryal et al. (2024)'s desirability axioms.

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1 Introduction

In many economic and social contexts, we encounter situations where objects or positions need to be ranked or ordered based on their perceived value or desirability. Examples include ranking universities, public schools, academic journals, job positions in a firm hierarchy, or even the relative prestige of various neighborhoods of a city. While such rankings are ubiquitous, the process of generating them in a systematic and economically meaningful way is far from trivial, and many of these current ranking systems suffer from well-known flaws.[2] From a theoretical perspective, many results in social choice theory speak to the difficulty of this problem (Arrow (1950); Gibbard (1973); Satterthwaite (1975)).

One approach for ranking objects is to use a market mechanism. In classical economic theory, competitive equilibrium prices serve as a measure of relative scarcity and desirability, and it may seem natural to use these prices to construct a social ranking. However, there are many scenarios where this may not work, either because these prices do not exist, or they are not a good indicator of desirability.[3]

Richter and Rubinstein (2015, hereafter, RR) generalize the classical notion of competitive equilibrium to a very broad class of abstract economies. We consider a setting in which each agent owns one indivisible object; following Shapley and Scarf (1974)'s classic model, we call these objects *houses*. In this setting, RR's notion of competitive equilibrium has rankings play the role of prices, and each agent's "budget set" consists of any object ranked weakly lower than her endowment. Each agent demands their most preferred house in their budget set, and a competitive equilibrium is a ranking and an allocation such that each agent receives their demand and the allocation is feasible.

RR interpret the competitive equilibrium ranking that comes out of their model as ordering the houses by their "prestige". An issue, however, is that competitive equilibrium rankings are not unique, and may indeed contradict each other with regard to which object is ranked higher than another. This is problematic for interpreting these rankings as a measure of prestige.

A different approach to constructing rankings is taken by Aryal et al. (2024, hereafter, AMT). Loosely stated, their approach is to first find a Pareto efficient, Pareto improvement of an assignment. They refer to this as a shadow assignment. The foundation of their ranking is

[2]See Toor (2000), Golden (2001), or Avery et al. (2013) for discussion of the influential *US News and World Report* rankings of colleges and universities; see Martin (2016), Fong and Wilhite (2017), Ioannidis and Thombs (2019) for academic journals; see Sbicca et al. (2012) or Jena et al. (2012) for rankings of medical residency programs.

[3]For instance, while colleges do charge a price (tuition), this is not a market-clearing price.

what they call *desirability.* If an agent prefers a house to her shadow assignment, then she desires it, and the preferred house should be ranked higher. Conversely, in order for a house to be ranked highly, it should be desired by an agent at a house ranked just below it. Any ranking that satisfies these two criteria is called a *desirable ranking.* As AMT show, when preferences over houses are determined as a combination of idiosyncratic preferences and a common quality component, desirable rankings succeed in ranking the houses by the true underlying quality as the market grows large.

The main contributions of this paper are: (1) to establish a fundamental connection between desirable rankings and competitive equilibrium rankings; (2) show how desirable rankings address the fundamental problem of competitive equilibrium rankings, namely their lack of uniqueness; and (3) introduce an algorithm, Delayed Trading Cycles (DTC) for finding the unique desirable, competitive equilibrium ranking. In particular, we give market interpretations of the desirability axioms of AMT, and show how to use one of AMT's axioms (the axiom of desire) to characterize the set of competitive equilibrium rankings. Then, we discuss the market interpretation of AMT's second axiom, which they call *justification.* In the market context, justification is akin to lack of inflation, where a ranking is said to be *inflated* if the rankings of some objects can be lowered without changing any agent's demand. We show that the no inflation condition is equivalent to justification. Using this condition, we show there is a unique ranking desirable ranking, and thus also a unique competitive equilibrium ranking without inflation. We call this ranking the *true prestige.*

Finally, we show how to calculate the true prestige using the DTC algorithm. Intuitively, this algorithm works by finding a set of colleges such that no student outside of the set desires any college in the set. These colleges are placed in the bottom tier, removed, and the process is repeated. Formally, we use the Top Trading Cycles (TTC) algorithm of Shapley and Scarf (1974) to identify these houses. It is well-known that the TTC outcome (i.e., the reallocation after all indicated trades are made) is independent of the order in which cycles are removed (the "cycle selection rule"). The typical implementation of TTC removes cycles immediately, as soon as they form. However, this discards information contained in the ordering of the cycles, which is relevant for constructing a ranking. In particular, we define a cycle to be a *last cycle* if there exists a cycle selection rule where said cycle is chosen last when running TTC. In general, such a cycle could be chosen earlier (and thus ranked higher) when running TTC. This multiplicity of cycle selection rules is what gives rise to the multiplicity of competitive equilibrium rankings, as each cycle selection rule will correspond to a different competitive equilibrium ranking. However, if a cycle is a last cycle, this means that no student outside of the cycle desires any house inside the cycle, which suggests that these houses are in low demand, and should actually be ranked low. The DTC algorithm proceeds by identifying the last cycles, ranking them last, removing, and repeating. As we show, this produces the unique desirable ranking, or, equivalently, the unique competitive equilibrium ranking with no inflation, i.e., the true prestige.

2 Preliminaries

An economy is a tuple $\mathcal{E} = (I, X, \omega, P)$, where $I = \{i_1 \ldots, i_N\}$ is a set of N individuals and $X = \{x_1, \ldots, x_N\}$ a set of of N indivisible objects; following Shapley and Scarf (1974), we will refer to the objects as *houses*. Each agent owns one house, given by the vector $\omega = (\omega_1, \ldots, \omega_N)$; we call ω_i agent i's **endowment**.

Agents may desire other houses more than the one they own. Let $P = (P_i)_{i \in I}$ be a profile of preferences, where P_i is a strict ranking over X. We write $x \; P_i \; y$ to denote that agent i strictly prefers house x to house y, and $x \; R_i \; y$ to denote either $x \; P_i \; y$ or $x = y$. An **allocation** is a function $\mu : I \to X$ that assigns each agent to a house, where μ_i is the house assigned to i. We say μ is **feasible** if $\mu_i \neq \mu_j$ for all $i \neq j$, i.e., each agent is assigned to a unique house.[4]

The goal is to provide a public ranking of the houses. A ranking is a weak ordering on the set X. We use $\trianglerighteq$ to denote a ranking, where $x \trianglerighteq y$ means that x is ranked weakly higher than y, and $x \triangleright y$ means that x is ranked strictly higher than y. Ties are allowed, i.e. both $x \trianglerighteq y$ and $y \trianglerighteq x$ may hold. This means that a ranking effectively divides the houses into *tiers*. We use the function $\tau(x)$ to denote the tier of house x, where $\tau(x) = 1$ means that $x \trianglerighteq y$ for all $y \in X$, $\tau(x) = 2$ means $x \trianglerighteq y$ for all $y \in X$ except those in tier 1, etc. We also use $\tau(i)$ to refer to the tier of agent i, which is the tier of the house i owns, i.e., $\tau(i) = \tau(\omega_i)$.

3 Competitive Equilibrium Rankings

In a classic economy, a competitive equilibrium is an allocation and prices such that the allocation is feasible and each agent receives their demand, given the prices. It may thus be natural to think of prices as signaling quality: higher-priced objects are more in demand, and so are of higher quality. One idea is then to construct a public ranking using these prices. Richter and Rubinstein (2015) generalize the classic competitive equilibrium concept to a much broader class of abstract economies where prices may not exist, but where a public ranking can play the "role" of price, and be used directly in the equilibrium definition.

[4] We use the notation μ for generic assignments that need not be equal to the endowment ω.

In the context of our model, RR's definition is as follows. Given a ranking $\trianglerighteq$ and an endowment vector ω, define agent i's *budget set* as $B_i(\trianglerighteq, \omega_i) = \{x : \omega_i \trianglerighteq x\}$ and their *demand* as $D_i(\trianglerighteq, \omega_i) = \max_{P_i}\{x : x \in B_i(\trianglerighteq, \omega_i)\}$. Further, we let $D(\trianglerighteq, \omega) = (D_i(\trianglerighteq, \omega_i))_{i \in I}$ be the vector of agent demands.[5]

Definition 1. *An **competitive equilibrium** is a pair $(\mu, \trianglerighteq)$ such that (i) μ is a feasible allocation and (ii) for all i, $\mu_i = D_i(\trianglerighteq, \omega_i)$.*

RR call $\trianglerighteq$ a **public ordering**, and interpret it as a social ordering that reflects the "prestige" of each of the houses. Notice how this is a generalization of the classic notion of a competitive equilibrium. The public ordering $\trianglerighteq$ plays the role of prices, with $x \triangleright y$ meaning that "x is more prestigious than y", or "x is more expensive than y". Then, the set $B_i(\trianglerighteq, \omega_i)$ is like agent i's "budget set" in the classic sense: it says that the agent can "afford" anything that is less prestigious (cheaper) than their endowment. An allocation is then an equilibrium in the standard sense: the allocation must be feasible and assign to each agent their most preferred house in their budget set, i.e., their favorite house that has prestige lower than (or equal to) their endowment.

For any economy $\mathcal{E}$, it is easy to construct a competitive equilibrium using Gale's top trading cycle argument (cf. Shapley and Scarf (1974)): simply run the top trading cycles (TTC) algorithm, and let the public ordering be the order in which cycles are removed. This is shown in the following example.

Example 1. Let there be 6 agents and 6 houses. The initial endowment profile is $\omega(i_k) = x_k$; i.e., agent i_k owns house x_k. The preferences are given as follows (where dots indicate that the remaining preferences are arbitrary).

i_1	i_2	i_3	i_4	i_5	i_6
x_2	x_1	x_4	x_3	x_1	x_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	x_5	x_6

Running Gale's top trading cycles algorithm, the initial top trading cycles are (x_1, x_2) and (x_3, x_4). After these are removed, the next set of top trading cycles is (x_5) and (x_6) (two

[5] A competitive equilibrium should not be interpreted literally as agents trading their assigned houses until equilibrium is reached; the competitive equilibrium "thought experiment" is used just to provide a public ranking of the houses.

self-cycles). Thus, the following is one example of a competitive equilibrium:[6]

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 \\ x_2 & x_1 & x_4 & x_3 & x_5 & x_6 \end{pmatrix} \qquad \trianglerighteq\colon \{x_1, x_2, x_3, x_4\} \triangleright \{x_5, x_6\}$$

For instance, agent i_1's endowment is x_1, and so their budget set is everything ordered weakly lower than x_1, which in this case, is all of the houses. Their favorite house in this set (i.e., their demand) is x_2. For agent i_5, who is endowed with x_5, the budget set is only $\{x_5, x_6\}$. So, while agent i_5 would prefer object x_1, this is out of their budget, and thus, i_5 demands x_5. It is easily checked that at the proposed ranking $\trianglerighteq$, the allocation μ gives each agent their most demanded house, and so $(\mu, \trianglerighteq)$ is a competitive equilibrium. ■

The top trading cycles argument shows that a competitive equilibrium in the sense of Definition 1 always exists.[7] However, as we will see later, the problem is that the competitive equilibrium public ordering need not be unique. For instance, in Example 1, we found one particular CE public ordering, but later we will show that there are many public orderings that are also CE (with the same allocation). This lack of uniqueness creates problems if we want to interpret the public ordering as a ranking of "prestige", as the multiple orders may be contradictory.

4 Desirable Rankings

RR interpret the ordering in a CE as representing prestige. Therefore, a CE provides a market-based ranking of the houses. AMT introduce a different approach for producing a ranking that they call *desirable rankings*. Their approach is based upon the idea that Pareto inefficiencies reflect idiosyncratic preferences. Therefore, Pareto improving an allocation acts as a filter to isolate what is common across the agents' preferences.

Definition 2. *Fix an initial matching μ. A matching μ^* is said to be a* ***shadow matching*** *(of μ) if μ^* is Pareto efficient and $\mu^*(i)\ R_i\ \mu(i)$ for all $i \in I$.*

In words, a shadow matching μ^* is a Pareto-efficient, Pareto improvement of the original matching μ.

[6] The notation $\trianglerighteq\colon \{x_1, x_2, x_3, x_4\} \triangleright \{x_5, x_6\}$ means that houses x_1, x_2, x_3, x_4 are all equivalent in the public ordering $\trianglerighteq$, and are ranked strictly higher than x_5 and x_6, with the latter two also being equivalent. We will use this convention throughout the rest of the paper as well.

[7] Shapley and Scarf (1974) show that a competitive equilibrium in the classic sense always exists: just set prices such that houses in earlier cycles have higher prices than those in later cycles. Roth and Postlewaite (1977) show that the allocation found by TTC is the unique competitive allocation.

Definition 3. *Let μ be an initial matching and μ^* be a shadow matching of μ. We say that student i **desires** a school c if $c \; P_i \; \mu^*(i)$. Student i **weakly desires** a school c if $c \; R_i \; \mu^*(i)$.*

Desire is both a strengthening of simple preferences and a way of isolating the common component of preferences.

Axiom 1 (Axiom of Desire)**.** A ranking $\trianglerighteq$ satisfies the **axiom of desire (AoD)** if there exists a shadow matching μ^* such that for every $i \in I$ and every $c \in C$ that i desires, we have $c \triangleright \mu^*(i)$.

AMT's second axiom is a converse to AoD.

Axiom 2 (Justification)**.** A ranking $\trianglerighteq$ is **justified** if for every tier $k < K$ and every college $c_k \in \Pi_k^{\trianglerighteq}$, there exists a sequence of colleges $c_k := c \triangleright c_{k+1} \triangleright \cdots \triangleright c_K$ and a sequence of students $i_{k+1}, \ldots, i_K$ such that $\mu^*(i_j) = c_j$ and student i_j desires c_{j-1} for all $j = k+1, \ldots, K$.

Intuitively, justification requires that in order for a house to be ranked k^{th}, it must be desired by an agent at a $(k+1)^{st}$-ranked house. An important technical point to note is that the justification is based on the agent's shadow assignment and not her original assignment. This equivalent formulation will be useful, so we state it as a lemma.

Lemma 1. *A ranking $\trianglerighteq$ is justified if and only if for every $k < K$ and every college $c \in \pi_k^{\trianglerighteq}$, there is a college $c' \in \pi_{k+1}^{\trianglerighteq}$ such that $\mu^*(c')$ desires c.*

Proof. Suppose $\trianglerighteq$ is a justified ranking. Consider any $k < K$ and any $c \in \pi_k$. By definition, there is a sequence of colleges $c_{k+1} \triangleright c_{k+2} \triangleright \ldots \triangleright c_K$ and a sequence of students $i_{k+1}, \ldots, i_K$ such that $\mu^*(i_j) = c_j$ and student i_j desires c_{j-1}. We have $K-k+1$ colleges, and $K-k+1$ partitions $(\pi_k, \ldots, \pi_K)$. Each college in the sequence must be in a lower partition then the next sequence. The only way this is possible is if for each k' in the sequence, $c_{k'} \in \pi_{k'}$. Therefore, indeed, $c_{k+1} \in \pi_{k+1}$ and $\mu^*(c_{k+1})$ desires c_k.

The reverse direction is also straightforward. Consider any $k < K$ and any $c \in \pi_k$. There is a college $c' \in \pi_{k+1}$ such that $\mu^*(c')$ desires c. Let $c' = c_2$. If $k+1 = K$, then we are done. If not, then there is a college $c_3 \in \pi_{k+2}$ such that $\mu^*(c_3)$ desires c_2. Repeating until we reach K, we have constructed a sequence $c \triangleright c_2 \triangleright c_3, \ldots \triangleright c_K$ such that $\mu(c_{\ell+1})$ desires c_ℓ. Therefore, $\trianglerighteq$ is justified. $\square$

5 Relationship between Competitive Orderings and Desirable Rankings

There is a close but not immediate relationship between an ordering in a CE and a desirable ranking. The next example shows that there is not a direct relationship between the two.

Example 2. Suppose there are three agents, $\{i_1, i_2, i_3\}$, and three houses, $\{A, B, C\}$. Agents i_1, i_2, and i_3 are allocated houses A, B, and C, respectively. The agents' preferences over houses are:

i_1	i_2	i_3
B	A	A
A	B	C
C	C	B

There is a unique Pareto improvement of the allocation: i_1, i_2, and i_3 are allocated houses B, A, and C, respectively. Therefore, this is the unique shadow assignment μ^*. Note that under μ^*, i_3 desires A but no agent desires B or C. Therefore, the unique desirable ranking is

$$[A] \triangleright [B, C].$$

The unique ordering for a CE is

$$[A, B] \triangleright [C].$$

A CE assignment is Pareto efficient. Therefore, i_1, i_2, and i_3 demand B, A, and C, respectively. We conclude that A and B must be the same price, and that C must be less expensive or else i_3 would also demand A.

AMT make no restriction on how the initial allocation of agents to houses is made. There are a number of economies, such as college admissions or academic journals, where the allocation is made through a competitive process of selection. In these, the "house" selecting the "agent" is analogous to price competition in a classical market. In the classical market, the requirement for an equilibrium is that goods that an agent prefers to her demand must be too expensive. The analogous condition for stability is that if a student prefers a college to her own, that college must have rejected her.

In this section, we consider markets where the selectivity of the house corresponds to the quality of the house. For expositional ease, we will refer to the agents and houses as students and colleges, respectively. If school A rejects a student while school B accepts the same student, this is some evidence that A is a better school than B. We introduce a third axiom to capture this intuition.

Axiom 3 (Selection)**.** A ranking $\trianglerighteq$, with shadow assignment μ^*, is **selective** if $\mu^*(i) \trianglerighteq \mu(i)$ for every agent i.

We will show that there is a direct connection between competitive equilibria and rankings that satisfy AoD and selection. To emphasize this connection, we will call a ranking competitive if it satisfies these two axioms.

Definition 4. *A ranking is* ***competitive*** *if it satisfies AoD and selection.*

Theorem 1. *A pair $(\mu^*, \trianglerighteq)$ is a competitive equilibrium if and only if $\trianglerighteq$ is a competitive ranking with shadow assignment μ^*.*

First, we establish that for a competitive ranking, a student i's assignment, $\mu(i)$, and shadow assignment, $\mu^*(i)$, must be ranked the same.

Lemma 2. *Let μ^* be a shadow assignment for competitive ranking $\trianglerighteq$. Then for every student i, $\mu(i) \simeq \mu^*(i)$.*

Proof. If $\mu^*(i) = \mu(i)$, then the result is trivial. Suppose there is a student i such that $\mu^*(i) \neq \mu(i)$. Construct a sequence of students and schools $i_1, c_1, i_2, c_d, \ldots$ by: (i) $i_1 = i$, (ii) $c_k = \mu^*(i_k)$, and (iii) $i_{k+1} = \mu(c_k)$. As there are only finitely many agents, this process must cycle. Further, by selection, for any student j, $\mu^*(j) \trianglerighteq \mu(j)$. Therefore, $c_1 \trianglerighteq c_2 \trianglerighteq c_3 \trianglerighteq \ldots$. As the colleges form a cycle, they must all be ranked the same. □

Notice that Lemma 2 establishes a direct market interpretation of the selectivity axiom. A selective ranking is analogous to a market where each agent spends her entire budget.

Proof of Theorem 1. Let $(\mu^*, \trianglerighteq)$ be a competitive equilibrium. First, we show that μ^* is a shadow assignment, which is to say, a Pareto efficient, Pareto improvement of μ. By definition, μ^* is feasible. A CE must be Pareto efficient. If there existed a Pareto improving trade, then one of the schools, s, must be the most expensive. The student currently "buying" s can already afford all of the schools involved in the trade; therefore, it contradicts that s is her demand if there is another school that she can afford and that she strictly prefers. Similarly, μ^* must be a Pareto improvement. Each student can afford her endowment. Therefore, her demand must be weakly preferred to her endowment.

Next, we show that $\trianglerighteq$ satisfies the axioms AoD and selectivity. To show that $\trianglerighteq$ is selective, we must show that for any student i, $\mu^*(i) \trianglerighteq \mu(i)$. Construct a sequence $i_1 = 1, c_1 = \mu^*(i_1), i_2 = \mu(c_1), c_2 = \mu^*(i_2), i_3 = \mu(c_2), \ldots$. Since i_k can afford her demand, for every k, $\mu(i_k) \trianglerighteq \mu^*(i_k)$. Therefore, $c_{k+1} \trianglerighteq c_k$, for all k. However, there are only finitely many colleges, so the sequence must cycle. Therefore, each college in the sequence must be ranked the same. So, indeed $\mu^*(i) \trianglerighteq \mu(i)$, and in fact, $\mu^*(i) \simeq \mu(i)$.

To show AoD, fix a student i, and consider any college c' that i desires: $c' \ P_i \ \mu^*(i)$. By the definition of demand, i must not be able to afford c: $c \triangleright \mu(i)$. We have already shown that $\mu^*(i) \simeq \mu(i)$; therefore, $c \triangleright \mu^*(i)$. But this is AoD: any school that a student strictly prefers to her shadow assignment must be ranked strictly higher than her shadow assignment.

The reverse direction is analogous. Suppose $\trianglerighteq$ is a competitive ranking with shadow assignment μ^*. By Lemma 2, for every student i, $\mu(i) \simeq \mu^*(i)$. Since $\trianglerighteq$ satisfies AoD, if $c' \ P_i \ \mu^*(i)$, i.e. i desires c', then $c' \triangleright \mu^*(i)$. Therefore, if $c' \ P_i \ \mu^*(i)$, then $c' \triangleright \mu(i)$. In words, i can afford $\mu^*(i)$ ($\mu(i)$ and $\mu^*(i)$ are ranked the same) and any college i strictly prefers to her shadow assignment is outside her budget set. Therefore, $\mu^*(i)$ is her favorite object that she can afford. Mathematically, $\mu^*(i) = D_i(\trianglerighteq, \mu)$. As μ^* is a feasible allocation, $(\mu^*, \trianglerighteq)$ is a competitive equilibrium. $\square$

5.1 The Weakness of Competitive Equilibria and Rankings

As discussed in Section 3, a competitive equilibrium always exists. However, a drawback is that it may not be unique. To see this, let us return to Example 1. Above, we found the following competitive equilibrium ranking:

$$\trianglerighteq: \{x_1, x_2, x_3, x_4\} \triangleright \{x_5, x_6\}$$

This was found using the TTC algorithm, and placing all houses removed at the same step in the same tier, and above any house removed in a later step. Here, the initial top trading cycles are (x_1, x_2) and (x_3, x_4), and so they are ranked first. After removing these houses, two self-cycles form, (x_5) and (x_6), and these are ranked together, in the second tier.

However, this is not the only competitive equilibrium ranking. For instance, all of the following are public orderings that, combined with the allocation μ from Example 1, are competitive equilibria:[8]

$$\{x_1, x_2, x_3, x_4\} \rhd \{x_5, x_6\}$$
$$\{x_1, x_2\} \rhd \{x_3, x_4\} \rhd \{x_5, x_6\}$$
$$\{x_3, x_4\} \rhd \{x_1, x_2\} \rhd \{x_5, x_6\}$$
$$\{x_1, x_2\} \rhd \{x_3, x_4\} \rhd \{x_5\} \rhd \{x_6\}$$
$$\{x_1, x_2\} \rhd \{x_3, x_4\} \rhd \{x_6\} \rhd \{x_5\}$$

While all of these rankings "support" the same allocation in the sense that any prices consistent with the above rankings will be competitive equilibria and thus stable, if we are to interpret the public ordering as "prestige", this is quite problematic: in some of the above orderings, x_1 is more prestigious than x_3, while in others it is the reverse; similarly, in some orderings x_5 is more prestigious than x_6, in others x_6 is more prestigious than x_5, while in still others they are ranked the same!

While the CE ranking is not unique, there is a unique competitive equilibrium allocation.

Proposition 1. *Let $\trianglerighteq$ be a competitive equilibrium ordering. Then, for every agent i, $D_i(\trianglerighteq, \omega_i) = \mu_i^{TTC}$, where μ^{TTC} is the allocation found by running the TTC algorithm starting with ω.*

Proof. Fix a CE ordering $\trianglerighteq$, and let

$$\Pi = (\pi_1, \ldots, \pi_K)$$

be the ordered partition induced by $\trianglerighteq$. Note that if $\omega_i \in \pi_1$, then i can afford any house. Therefore, i's demand is her favorite house. In TTC, an agent initially points at her favorite house. If $\omega_i \in \Pi_1$, then i points at $D_i(\trianglerighteq, \omega_i)$. As demand is a feasible assignment, if $h \in \pi_1$, then there must be some agent i who demands h. As h is the most expensive house, $\omega_i \in \pi_1$ or else i could not afford h. By the pigeonhole principle, every agent whose endowment is in π_1 demands a house in π_1 or else demand would not be feasible. So, indeed, we can decompose π_1 into top trading cycles.

The remaining argument is identical. Consider the cycle selection rule for TTC where in the first round we remove all the cycles in π_1. The same argument as above shows that if agent i's endowment is in π_2, then i's demand is her favorite house (except possibly for houses in π_1). Further, all houses in π_2 must be demanded by someone or else demand would not be a feasible

[8]This same example exhibits why we cannot use competitive equilibrium prices in the classical sense to construct a ranking of the houses: while there is a unique competitive equilibrium allocation (cf. footnote 7), the price vector is not unique: any price vector consistent with any of the orderings below will be a competitive equilibrium.

assignment. Therefore, we can decompose π_2 into top trading cycles. Repeating this argument, we get the desired result. $\square$

For each house h, running TTC determines a unique set of houses that h forms a top trading cycle with. Given Proposition 1, this set will be important for our analysis. Therefore, we explicitly define it.

Definition 5. *Given a house h, h's* ***trading cycle****, denoted $[h]$, is the set of houses that form a top trading cycle with h.*

Corollary 1. *For any house x, any CE $(\mu, \trianglerighteq)$, and any $y \in [x]$, we have $y \simeq x$.*

6 Achieving Uniqueness: The True Prestige

If we are to interpret a CE ranking as a measure of prestige, the ranking should be unique, or else the interpretation does not make sense. In this section, we (i) propose a refinement of CE rankings (ii) show that this refinement produces a unique CE ranking and (iii) discuss the relationship between this refinement and the justification axiom of AMT.

Consider a classic demand setting with prices p and demand functions $z_i(p)$ for each agent i. If we raise the prices of a set of goods and this does not change what anyone purchases, this can be interpreted as "inflation". In other words, if there are two price vectors p and p' such that (i) $p' \geq p$ and (ii) for every agent i, $z_i(p') = z_i(p)$, then p' *inflates* p.

In our model, the ranking $\trianglerighteq$—which can be interpreted as prestige—plays the role of prices, and so we need a notion of inflation for our setting. Note that there is another way to view the definition of inflation just introduced: rather than starting at p and raising the price vector to p', we could start at p' and lower the price vector to p, without changing the demand. After lowering the price vector, some goods that were once more expensive than another good are now cheaper than it. For a general order $\trianglerighteq$, this is akin to lowering the ranking of a set of goods.

First, we define "lowering prices" for a general linear order. Given a ranking $\trianglerighteq$, let $L^{\trianglerighteq}(x) = \{y \in X | x \triangleright y\}$. In words, $L^{\trianglerighteq}(x)$ is the set of all goods that are ranked strictly below x in ordering $\trianglerighteq$. Intuitively, the prestige of a school s is lowered if it is moved to a lower partition. Equivalently, the prestige of s has lowered if the set of schools ranked strictly below it is a strict subset of the prior set. This is the foundation for our definition of "lowering the price" on a set of goods.

Definition 6. *Fix a ranking $\trianglerighteq$. Ranking $\trianglerighteq'$* ***lowers the prestige*** *of a set of goods $X' \subseteq X$ if (i) for every $x \in X'$, $L^{\trianglerighteq'}(x) \subsetneq L^{\trianglerighteq}(x)$ and (ii) for $y, z \notin X'$, $y \trianglerighteq z \iff y \trianglerighteq' z$.*

The following example is intended to make this abstract notion more concrete.

Example 3. Consider the following orderings of schools $\{a, b, c, d, e, f\}$. For illustration, we assign specific numbers to represent prices, but of course, this choice is arbitrary.

	4	3	2	1	*prestige lowered?*
π^1	[a,b,c]		[d,e]	[f]	
π^2	[a,b]	[c]	[d,e]	[f]	no
π^3	[a,b]		[c,d,e]	[f]	yes
π^4	[a]		[b,c,d,e]	[f]	yes
π^5	[a,b]		[c,d]	[e,f]	yes
π^6	[a,b]		[d,e]	[c,f]	yes
π^7	[a,b]		[d]	[c,e,f]	yes

- Π^2: Visually, it looks like π^2 lowers the prestige of c. However, this is not our interpretation (and does not meet our definition). The set of goods ranked strictly below c is the same in π^1 and π^2: $\{d, e, f\}$. Under the price interpretation, every agent who wishes to buy c under one price can still buy c under the other. We interpret π^2 as increasing the price (prestige) of a and b (strictly fewer agents can buy a or b under π^2) and now as lowering the price (prestige) of c.

- π^3: Under π^1, the set of student ranked strictly below c is $\{d, e, f\}$ while under π^3 the set is $\{f\}$. So yes, π^3 lowers the prestige of c.

- π^4: Yes, the prestige of two schools, b and c, has been lowered.

- π^5: Strictly fewer schools are ranked below c and e, respectively. The relative ordering of the other schools has not changed, so yes, π^5 lowers the prestige of schools $X = \{c, e\}$.

- π^6: Yes, the prestige can drop more than one level.

- π^7: $L^1(c) = \{d, e, f\}$ and $L^7(c) = \emptyset$. $L^7(e) = \{f\}$ and $L^7(e) = \emptyset$. The relative ordering of the other schools has not changed. Therefore, π^7 lowers the prestige of $X = \{c, e\}$. Notice that the student assigned to e can now afford c. So it is possible that a school's prestige is lowered, but its owner's budget set expands.[9]

[9]This also occurs with standard prices. Goods 1 and 2 can initially have prices of 7 and 5, respectively, and both prices can be lowered to 3. The owner of good 2 can now afford good 1, but we still say the price of both goods has been lowered.

With this definition, we can now define inflation. If we lower the prestige for a set of schools, then by definition, there are students who can now afford these schools but previously could not. If the demand for all students does not change, then we interpret the previous prestige as being inflated.

Definition 7. *Ranking $\trianglerighteq$* ***inflates prestige*** *if there exists a set of houses $X' \subseteq X$ and another ranking $\trianglerighteq'$ such that (i) $\trianglerighteq'$ lowers the prestige of X' and (ii) for every agent i, $D_i(\trianglerighteq', \omega_i) = D_i(\trianglerighteq, \omega_i)$.*

Definition 8. *Ranking $\trianglerighteq$ is the* ***true prestige*** *if (i) $\trianglerighteq$ is a CE ranking and (ii) $\trianglerighteq$ does not inflate the prestige of any set of houses.*

Of course, it would not make sense to speak of the true prestige if it were not unique. This will be the content of the next theorem. However, before stating that theorem, we return to Example 1 to give a better understanding of inflated rankings and the true prestige. Above, we exhibited several rankings that are CE rankings. Consider first the ranking

$$\{x_1, x_2\} \triangleright \{x_3, x_4\} \triangleright \{x_5, x_6\} \tag{1}$$

Notice that this ranking inflates the prestige of x_1 and x_2. To see this, note that under this ranking, the alternative ranking $\trianglerighteq'$

$$\{x_1, x_2, x_3, x_4\} \triangleright' \{x_5, x_6\} \tag{2}$$

lowers the prestige of $\{x_1, x_2\}$ without changing any agent demands.

Ranking $\trianglerighteq'$, on the other hand, does not inflate the prestige of any house. For instance, if we lowered the prestige of x_1 (x_3) by placing it in the same tier as x_5 and x_6, this would change the demand of agent i_5 (i_6), who can now afford x_1 (x_3). If we lowered the prestige of x_2 (x_4), this would change the demand of agent i_2 (i_4), who would no longer be able to afford x_1 (x_3). Thus, ranking $\trianglerighteq'$ does not inflate the prestige of any house.

Theorem 2. *There exists a unique CE ranking that does not inflate the prestige of any set of houses.*

We prove Theorem 2 constructively by defining an algorithm that finds a CE with no inflation and using the the properties of this algorithm to prove uniqueness. Therefore, we introduce this algorithm before proving Theorem 2.

6.1 Delayed Trading Cycles

In this section, we present an algorithm for finding a CE with no inflation. We call this algorithm Delayed Trading Cycles (DTC). In a typical use-case of TTC (such as in school choice settings, e.g., Abdulkadiroğlu and Sönmez (2003)), the trades indicated by the cycles are implemented to reach a Pareto efficient (re-)matching. It is well-known that for these purposes, the order in which cycles are removed in TTC is irrelevant: cycles can all be removed immediately as they are formed, or they can be removed one at a time, in any order, without affecting the final matching. Typically, all top trading cycles are removed immediately as they are formed; however, doing so discards important information contained in the cycle selection order. The delayed trading cycles algorithm will make use of this information to construct the ranking.

The key feature of top trading cycles we will use is that once a cycle is formed, it remains a cycle until it is removed. Thus, for some cycles, it is possible to let them remain, and proceed using a cycle selection order that removes other cycles first. In some cases, it may even be possible to choose a selection order that removes all other cycles before a given cycle χ; when this is true, we call χ a *last cycle*. For other cycles χ', it will not be possible to do this. This happens when students from outside of χ' are pointing to schools inside χ': since these students continue to point into χ' until χ' is removed, it is not possible for χ' to be a last cycle. In other words, last cycles are cycles such that no students from outside of the cycle are pointing to schools inside the cycle. Last cycles thus contain the least prestigious schools, and the algorithm works from the bottom up: we identify the last cycles and rank them last. These schools and students are then removed, and we identify the new set of last cycles on the remaining submarket. These schools are ranked one level higher (second-to-last). This process is repeated until all schools have been ranked.

We first provide the formal definitions, and then give an example of the algorithm.

Definition 9. *Given an outcome μ, let μ^{TTC} be the matching found by implementing all trades in the cycles when running the TTC mechanism. A set of colleges $\chi \subseteq C$ is a* ***last cycle*** *if χ is a cycle that forms in the TTC algorithm and it satisfies the following (equivalent) conditions:*

(i) There exists a cycle selection ordering such that cycle χ is chosen last.

(ii) For every student i such that $\mu(i) \notin \chi$, $\mu^{TTC}(i)\ P_i\ c'$ for every $c' \in \chi$.

Definition 10 (Delayed Trading Cycles (DTC) algorithm)**.** *Given an outcome μ, let μ^{TTC} be the outcome obtained by running TTC on μ. Recursively define the sets C^ℓ as follows:*

- *Step $\ell = 1$: C^1 is the set of last-cycles.*

- *Step ℓ: If $C \setminus \cup_{\ell'=1}^{\ell-1} C^{\ell'} \neq \emptyset$, then C^{ℓ} is the set of last-cycles of $C \setminus \cup_{\ell'=1}^{\ell-1} C^{\ell'}$. Otherwise, stop.*

*Let $C^1, \ldots, C^L$ be the resulting partition of the colleges. The **DTC ranking** of the colleges, $\trianglerighteq^{DTC}$, is given as follows: for any two colleges $a, b \in C$, where $a \in C^{\ell}$ and $b \in C^{\ell'}$, $a \trianglerighteq^{DTC} b$ if and only if $\ell \geq \ell'$. The tier-k colleges are $\Pi_k^{DTC} = C^{L-k+1}$.*

We now give an example to show how the DTC algorithm works.

Example 4. There are six students $I = \{i_1, i_2, i_3, i_4, i_5, i_6\}$ and six colleges $C = \{A, B, C, D, E, F\}$. The boxes indicate the initial matching μ, and the student preferences are shown below.

P_1	P_2	P_3	P_4	P_5	P_6
A	C	B	$A*$	$B*$	$\boxed{F*}$
B	A	$\boxed{C*}$	$\boxed{B}$	$\boxed{A}$	A
$D*$	$E*$	A	C	C	B
$\boxed{E}$	$\boxed{D}$	D	D	E	C
C	B	E	E	D	D
F	F	F	F	F	E

We begin by first running the TTC algorithm on μ. The initial top trading cycles are (A, B) and (F) (the latter is a self-cycle).[10] If we follow the typical implemenation of TTC, we remove these cycles, and the next top trading cycle is (C) (another self-cycle). Finally, removing this cycle, the remaining top trading cycle is (D, E). Implementing these trades results in the TTC assignment μ^{TTC}, which is the one indicated by the stars in the table.

Now that we have determined the cycles—$(A, B), (C), (D, E), (F)$—we proceed to create the ranking by identifying the last cycles. Notice that in the first step, the initial cycles are (A, B) and (F). The latter is a last cycle, while the former is not. To see this, note that it is possible to complete the algorithm by removing cycles in the order (A, B), (C), (D, E), (F), and so (F) is a last cycle. The cycle (A, B), on the other hand, is not a last cycle: we could instead begin by removing (F) first, but after this, the only cycle that is left is the cycle (A, B). The algorithm cannot continue until (A, B) is removed, and so (A, B) is a not a last cycle.

Equivalently, using part (ii) of Definition 9, notice that for cycle (F), for all students outside of the cycle—i.e., for all $i \neq i_6$—we have $\mu^{TTC}(i)\ P_i\ F$. For cycle (A, B) this does not hold:

[10] For brevity, we omit the students from the cycles, and write them as containing only the schools. Equivalently, we can define the graph such that each school c points to the top-ranked school of $\mu(c)$, its assigned student (which may or may not be school c itself).

for instance, for student i_1, $A \; P_1 \; \mu^{TTC}(i_1) = D$. Thus, according to part (ii) of Definition 9, (A, B) is not a last cycle.

It turns out that the cycle (D, E) is also a last cycle: the cycle selection ordering $(A, B), (C)$, $(F), (D, E)$ is also a possible implemenation of the TTC algorithm. Cycle (C) is not a last cycle. Thus, the last cycles in step 1 of the DTC algorithm are (D, E) and (F). These schools are ranked last, and then they and their students are removed. After this, we are left with the cycles (A, B) and (C), for which (C) now is a last cycle. We remove school C and rank it one step ahead of D, E, and F. Finally, the only remaining cycle is (A, B), which is ranked highest. The final ranking is

$$\trianglerighteq^{DTC}: \{A, B\} \triangleright \{C\} \triangleright \{D, E, F\}$$

6.2 Proof of Theorem 2

First, we show existence: DTC produces a CE with no inflation.

Proposition 2. *For any economy $\mathcal{E}$, let μ^{TTC} be the assignment produced by TTC and $\trianglerighteq^{TTC}$ be the ordering produced by DTC. Then $(\mu^*, \trianglerighteq^{TTC})$ is a CE with no inflation.*

Proof. Shapley and Scarf (1974) establish that $(\mu^*, \trianglerighteq^{TTC})$ is a CE. Therefore, we need to show that there is no inflation. Let

$$\Pi = (\pi_1, \ldots, \pi_K)$$

be the ordering produced by DTC.

Suppose, for contradiction, that there is a set of houses X and prices π' such that π' lowers the prestige of the houses in X but does not change the demand of any agent. Let h be the lowest-priced house in X (relative to prices π). First, note that the rank of h is not last, as the prestige of a house ranked last cannot be lowered. Suppose $h \in \pi_k$ where $k < K$.

As a reminder, $[h]$ denotes the houses in h's TTC cycle. First, we claim that $[h] \subset X$. If $[h] = h$, then this is true trivially. Otherwise, let i be the owner of h, and let h_2 be the house i is pointing to in the TTC cycle. If h's price is lowered, but h_2 is not, then i can no longer afford h_2, and i's demand would change. By assumption, no agents demand changes, therefore $h_2 \in X$, and so on.

We use the second, equivalent definition of a last cycle: a set of colleges χ is a last cycle if, for every student i such that $\mu(i) \notin \chi$, $\mu^{TTC}(i) \; P_i \; c'$ for every $c' \in \chi$. By construction, $[h]$ is a last cycle when houses $\pi_{k+1} \cup \ldots \pi_K$ are removed, but $[h]$ is not a last cycle of $\pi_1 \cup \ldots \pi_k \cup \pi_{k+1}$.

Therefore, there is a student i, assigned to a house in π_{k+1}, and a house $h' \in [h]$ such that $h' \; P_i \; \mu^{TTC}(i)$. Under π', there are strictly fewer houses ranked strictly below the houses in $[h]$. Since the ordering of the houses in $\pi_{k+1} \cup \ldots \pi_K$ has not changed (none of these houses are in X), the houses in π_{k+1} must be ranked the same or above the houses in $[h]$. Therefore, i can now afford h', and her demand changes under π'. This is a contradiction since by assumption, π' lowers the prices in X without changing any student's demand. □

Proposition 3. *For any economy $\mathcal{E}$, let $(\mu^*, \trianglerighteq)$ be a CE with no inflation. Then $\mu = \mu^{TTC}$, the assignment produced by TTC, and $\trianglerighteq$ is the ordering produced by DTC.*

Proof. Let μ^{TTC} be the assignment made by TTC, let $\trianglerighteq$ be the ordering produced by DTC, and let

$$\Pi = (\pi_1, \ldots, \pi_K)$$

be the partitions induced by $\trianglerighteq$. Let $(\mu^*, \trianglerighteq')$ be any CE with no inflation, and let

$$\Pi' = (\pi'_1, \ldots, \pi'_L)$$

be the partitions induced by $\trianglerighteq'$.

As a reminder, for any house h, we let $[h]$ denote the set of houses in h's top trading cycle. Further, by Corollary 1, for any house $h \in \pi'_k$, $[h] \subseteq \pi'_k$.

We will prove by induction that for all $0 \leq k < K$, that $\pi_{K-k} = \pi'_{L-k}$. For the base case, consider $k = 0$. By construction, the partition π_K consists of last cycles. Consider any college $c \in \pi_K$, and let $[c]$ be its associated trading cycle. As $[c]$ is a last cycle, for any student i such that $\mu^{TTC}(i) \notin [c]$, $\mu^{TTC}(i) \; P_i \; c'$ for every $c' \in [c]$. In words, not student outside of this cycle would buy any college in this cycle even if they were the lowest priced colleges. This means that if $c \notin \pi'_L$, then we can move $[c]$ to π'_L without changing the demand of any agent. Therefore, the prices of $[c]$ were inflated. As there is no inflation, we conclude that $[c] \subseteq \pi'_L$ and that $\pi_K \subseteq \pi'_L$.

Consider a college $c \in \pi'_L$. We know that $[c] \subset \pi'_L$. We need to show that $[c]$ is a last cycle. Any agent can afford any college in $[c]$. If agent i's demand is not in $[c]$, then she must strictly prefer her demand to any college in $[c]$. Mathematically, if $\mu^{TTC}(i) \notin [c]$, then $\mu^{TTC}(i) \; P_i \; c'$ for every $c' \in [c]$. This implies that $[c]$ is a last cycle, and we conclude that $\pi'_L \subseteq \pi_K$.

The inductive step is similar. Consider any $k > 0$ and suppose that $\pi_{k'} = \pi'_{k'}$ for all $k < k' \leq K$. Consider college $c \in \pi_{K-k}$. By construction, π_{K-k} is the union of last cycles when the colleges $\pi_{k+1} \cup \ldots \cup \pi_K$ are removed. If $c \notin \pi_{L-k}$, then we could lower $[c]$ to π_{L-k} without changing the demand of any agent. This would mean that π' has inflated prices, which would be a contradiction. Therefore, $\pi_{K-k} \subset \pi_{L-k}$. Now consider any college $c \in \pi'_{L-k}$. For any agent i, if $\mu(i) \in \pi'_1 \cup \ldots \pi'_{L-k}$, then i can afford every college in $[c]$. Therefore, either $\mu'(i) \in [c]$ or else $\mu'(i)\ P_i\ c'$ for every college $c' \in [c]$. As a result, when choosing cycles for TTC, so long as $(\pi'_1 \cup \ldots \pi'_{L-k}) \setminus [c] \neq \emptyset$, we can find a cycle other than $[c]$ to remove. We conclude that $[c]$ is a last cycle and that $\pi'_{L-k} \subseteq \pi_{K-k}$. $\square$

Theorem 2 is a direct consequence of Propositions 2 and 3. We show that there is a natural relationship between justification and inflation. First, the next example shows that there may not be a competitive ranking that is justified.

Example 5. We use the same economy as in Example 2. There are three agents, $\{i_1, i_2, i_3\}$, and three houses, $\{A, B, C\}$. Agents i_1, i_2, and i_3 are allocated houses A, B, and C, respectively. The agents' preferences over houses are:

i_1	i_2	i_3
B	A	A
A	B	C
C	C	B

There is a unique Pareto improvement of the allocation: i_1, i_2, and i_3 are allocated houses B, A, and C, respectively. Therefore, this is the unique shadow assignment μ^*. If $\trianglerighteq$ is a connected ranking, then $\mu^*(i_1) = B \trianglerighteq \mu(i_1) = A$ and $\mu^*(i_2) = A \trianglerighteq \mu(i_2) = B$. Therefore, in any connected ranking, $A \simeq B$. As i_3 desires A, the unique connected ranking satisfying AoD is $A \simeq B \triangleright C$. However, this ranking is not justified as there is no agent who desires B.

The justification axiom requires that every school not ranked last be desired by a student one tier below it. For a competitive ranking, for every college c, the schools in c's trading cycle, $[c]$, must be ranked the same as c. Therefore, a weaker but analogous justification condition for competitive rankings is that for every college not ranked last, a college in c's trading cycle must be desired by a student one tier below it.

Axiom 4 (Justification)**.** Given a ranking $\trianglerighteq$ with shadow assignment μ^*, $\trianglerighteq$ is **competitively justified** if for every tier $k < K$ and every college $c \in \Pi_k^{\trianglerighteq}$, there exists a student i such that (i) $\mu^*(i) \in \pi_{k+1}$ and (ii) $c'\ P_i\ \mu^*(i)$ for some $c' \in [c]$.

We show that for competitive rankings and competitive equilibria, the competitively justified axiom is equivalent to no inflation. This is an immediate corollary of our next theorem, which says that not only does there always exist a competitive ranking that is competitively justified, but it is unique.

Theorem 3. *Suppose each house has a capacity of one. A competitive ranking is competitively justified if and only if it is the ranking produced by DTC.*

Since the DTC ranking is the unique CE with no inflation, the following is an immediate corollary.

Corollary 2. *For any economy $\mathcal{E}$, $(\mu, \trianglerighteq)$ is a CE with no inflation if and only if $\trianglerighteq$ is a desirable, selective, and competitively justified ranking.*

Proof. Let μ^* and $\trianglerighteq$ be the DTC assignment and ordering, respectively. As DTC is an ordering of Top Trading Cycles, $(\mu^*, \trianglerighteq)$ is a competitive equilibrium; therefore, by Theorem 1, $\trianglerighteq$ is a competitive ranking with shadow assignment μ^*. We need to show that $\trianglerighteq$ is competitively justified. Consider any π_k where $k < K$, i.e. colleges that are not ranked last. For any college $c \in \pi_k$, by construction $[c]$ is not a last cycle in economy $\pi_1 \cup \dots \pi_{k+1}$ but is a last cycle in economy $\pi_1 \cup \dots \pi_k$. Therefore, there is a student i such that $\mu^*(i) \in \pi_{k+1}$, but i strictly prefers a college in $[c]$. Thus, the rankings of the colleges $[c]$ are competitively justified.

For uniqueness, let $\trianglerighteq'$ be any desirable, selective, and competitively justified ranking. By Proposition 1, the shadow assignment for $\trianglerighteq'$ is μ^*, the assignment made by TTC.

Let

$$\Pi = (\pi_1, \dots, \pi_K)$$

be the partitions induced by $\trianglerighteq$, and let

$$\Pi' = (\pi'_1, \dots, \pi'_L)$$

be the partitions induced by $\trianglerighteq'$. Suppose for contradiction that $\Pi \neq \Pi'$, and let k be minimal such that $\pi_{K-k} \neq \pi'_{L-k}$.

Let $c \in \pi_{K-k}$, $[c]$ is a last-cycle when the lower-ranked colleges are removed. Consider a college $c' \trianglerighteq c$ such that $c' \notin [c]$, and let $i = \mu^*(c')$. Since the colleges ranked below c are the same under $\trianglerighteq$ and $\trianglerighteq'$, $c' \trianglerighteq' c$. But since $\trianglerighteq'$ is a desirable ranking, $c' \; P_i \; c$. To summarize, for any student i such that $\mu^*(i)$ is ranked $K - k$ or higher and $\mu^*(i) \notin [c]$, i does not desire c (or any college in $[c]$). Therefore, it cannot be competitively justified to rank c higher than π_{L-k}. Since $\trianglerighteq'$ is competitively justified, $[c] \subseteq \pi_{L-k}$. This shows that $\pi_{K-k} \subseteq \pi_{L-k}$.

Similarly, consider any college $c \in \pi'_{L-k}$. As a partition must contain a trading cycle, $[c] \subseteq \pi'_{L-k}$. Let i be a student such that $\mu^*(i) \notin [c]$ and $\mu^*(i) \trianglerighteq' c$. Since $\trianglerighteq'$ satisfies AoD, i cannot desire any college in $[c]$. This implies that $[c]$ is a last cycle when the cycles $\pi_{L-k+1} \cup \ldots \pi_L = \pi_{K-k+1} \cup \ldots \pi_K$ are removed: any remaining student not in this cycle does not desire any college in the cycle. Therefore, $c \in \pi_{K-k}$ and we conclude that $\pi_{L-k} \subseteq \pi_{K-k}$.

We have found our contradiction. We assumed that $\pi_{K-k} \neq \pi'_{L-k}$ but have shown that $\pi_{K-k} = \pi_{L-k}$. □

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